A NEW INVARIANT OF G_2-STRUCTURES

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Abstract. We define a $\mathbb{Z}_{48}$-valued homotopy invariant $\nu(\varphi)$ of a G_2-structure φ on the tangent bundle of a closed 7-manifold in terms of the signature and Euler characteristic of a coboundary with a $Spin(7)$-structure. For manifolds of holonomy G_2 obtained by the twisted connected sum construction, the associated torsion-free G_2-structure always has $\nu(\varphi) = 24$. Some holonomy G_2 examples constructed by Joyce by desingularising orbifolds have odd ν. If M is 2-connected and the greatest divisor of $p_1(M)$ modulo torsion divides 224 then ν determines a G_2-structure up to homotopy and diffeomorphism; this sufficient condition is satisfied for many twisted connected sum G_2-manifolds.

1. Introduction

In this paper we develop methods to determine when two G_2-structures on a closed 7-manifold are deformation-equivalent, *i.e.* equivalent under homotopies through G_2-structures and diffeomorphisms. The main motivation is to study the problem of deformation-equivalence of metrics with holonomy G_2. Such metrics can be defined in terms of torsion-free G_2-structures. The torsion-free condition is a complicated PDE, but we ignore that and consider only the G_2-structure as a topological residue of the holonomy G_2 metric.

1.1. The ν-invariant.

A G_2-structure on a 7-manifold M is a reduction of the structure group of the frame bundle of M to the exceptional Lie group G_2. As we review in §2.1, a G_2-structure on M is equivalent to a 3-form $\varphi \in \Omega^3(M)$ of a certain type and we will therefore refer to such 'positive' 3-forms as G_2-structures. A G_2-structure induces a Riemannian metric and spin structure on M. Throughout this introduction M shall be a closed connected spin 7-manifold and all G_2-structures φ will be compatible with the chosen spin structure.

Two G_2-structures are homotopic if they can be connected by a continuous path of G_2-structures and we define

$$\mathcal{G}_2^h(M) := \{[\varphi]\}$$

to be the set of homotopy classes of G_2-structures on M.

Lemma 1.1. *The group $H^7(M;\pi_7(S^7)) \cong \mathbb{Z}$ acts freely and transitively on $\mathcal{G}_2^h(M) \equiv \mathbb{Z}$.*

While Lemma 1.1 is a basic fact of G_2-topology, it is often of greater interest to study G_2-structures up to *deformation*, where a deformation between G_2-structures φ and φ' is a spin diffeomorphism $f\colon M \cong M$ together with a homotopy from φ to the pull-back form $f^*\varphi'$. We define

$$\mathcal{G}_2^d(M) := \{[[\varphi]]\}$$

to be the set of deformation classes of G_2-structures on M. Up until now neither invariants of $\mathcal{G}_2^d(M)$ nor results about its cardinality have appeared in the literature. Our starting point for studying both of these problems is the following characteristic class formula, valid for any closed spin 8-manifold X (see Corollary 2.4):

$$e_+(X) = 24\widehat{A}(X) + \frac{\chi(X) - 3\sigma(X)}{2}. \tag{1}$$

Here the terms are the integral of the Euler class of the positive spinor bundle, the $\widehat{A}$-genus, the signature and ordinary Euler characteristic of X ($\widehat{A}(X)$ is an integer because X is spin, and $\sigma(X) \equiv \chi(X) \bmod 2$ for any closed oriented X). Moving from $Spin(8)$ to $Spin(7)$, if we use the (real dimension 8) spin representation of $Spin(7)$ to regard $Spin(7)$ as a subgroup of $GL(8,\mathbb{R})$, then a $Spin(7)$-structure on an 8-manifold X can be characterised by a certain kind of 4-form

$\psi \in \Omega^4(X)$. A $Spin(7)$-structure defines a spin structure and Riemannian metric on X, and (up to a sign) a unit spinor field of positive chirality. In particular, if a closed 8-manifold X has a $Spin(7)$-structure then $e_+(X) = 0$, and (1) implies

$$48\widehat{A}(X) + \chi(X) - 3\sigma(X) = 0. \tag{2}$$

If W is a compact 8-manifold with boundary M then a $Spin(7)$-structure on W induces a G_2-structure on M. From (2) one deduces that the "$\widehat{A}$ defect" $\chi(W) - 3\sigma(W)$ mod 48 depends only on the induced G_2-structure on M. It turns out, see Lemma 3.4, that any G_2-structure φ on any closed 7-manifold bounds a $Spin(7)$-structure on some compact 8-manifold and this allows us to define an invariant $\nu(\varphi)$.

Definition 1.2. Let (M, φ) be a closed connected spin 7-manifold with G_2-structure and $Spin(7)$-coboundary (W, ψ). The ν-invariant of φ is the residue

$$\nu(\varphi) := \chi(W) - 3\sigma(W) \mod 48 \ \in \mathbb{Z}_{48}.$$

Remark 1.3. Among the many analogous invariants in differential topology, perhaps the one best known to non-topologists is Milnor's $\mathbb{Z}_7$-valued λ-invariant of homotopy 7-spheres, defined as a "p_2 defect" of a spin coboundary [17].

Theorem 1.4 below summarises the basic properties of ν. Note that if φ is a G_2-structure on M, then the 3-form $-\varphi$ is also a G_2-structure, but compatible with the *opposite* orientation; $-\varphi$ is a G_2-structure on $-M$. In addition, if X is a closed $(2n+1)$-manifold, we define its rational semi-characteristic by $\chi_{\mathbb{Q}}(X) := \sum_{i=0}^{n} b^i(X)$ mod 2.

Theorem 1.4. *For all G_2-structures φ on M, $\nu(\varphi) \in \mathbb{Z}_{48}$ is well-defined, and invariant under homotopies and diffeomorphisms. Hence ν defines a function*

$$\nu : \mathcal{G}_2^d(M) \to \mathbb{Z}_{48}. \tag{3}$$

Moreover $\nu(-\varphi) = -\nu(\varphi)$, and ν takes all 24 values allowed by the parity constraint

$$\nu(\varphi) \equiv \chi_{\mathbb{Q}}(M) \mod 2. \tag{4}$$

Theorem 1.4 entails that $\mathcal{G}_2^d(M)$ has at least 24 elements. This leads to four interesting questions; two geometric in nature and the other two topological.

(i) What are the values of ν for G_2-structures arising from G_2 holonomy metrics? In particular do such metrics constrain the possible values of ν?
(ii) What is the cardinality of $\mathcal{G}_2^d(M)$? For example, for which closed spin manifolds M is ν a complete invariant of $\mathcal{G}_2^d(M)$?

We shall address these questions in turn in §1.3 and §1.4 below but first we will indicate how ν is related to Lemma 1.1 by interpreting G_2-structures in terms of spinor fields. Indeed spinor fields play an important role in the proof of Theorem 1.4. However, from the definition above we see that computing ν only requires the construction of a coboundary with the right type of 4-form, and finding such 4-forms is typically easier than describing spinor fields.

Example 1.5. S^7 has a standard G_2-structure φ_{rd}, induced as the boundary of B^8 with a flat $Spin(7)$-structure. Clearly $\nu(\varphi_{rd}) \equiv \chi(B^8) - 3\sigma(B^8) \equiv 1$. On the other hand, the flat $Spin(7)$-structure on the complement of $B^8 \subset \mathbb{R}^8$ induces the G_2-structure $-\varphi_{rd}$ on S^7 (with the orientation reversed). If r is a reflection of S^7 then $\bar{\varphi}_{rd} = r^*(-\varphi_{rd})$ is a different G_2-structure on S^7 inducing the same orientation as φ_{rd}. Since $\nu(\bar{\varphi}_{rd}) = \nu(-\varphi_{rd}) = -\nu(\varphi_{rd}) = -1$ there can be no homotopy between φ_{rd} and $\bar{\varphi}_{rd}$, which is a warning sign that we need to be careful about orientations.

Example 1.6. S^7 has a 'squashed' G_2-structure φ_{sq} that is invariant under $Sp(2)Sp(1)$ and nearly parallel (*i.e.* the corresponding cone metric on $\mathbb{R} \times S^7$ has exceptional holonomy $Spin(7)$). This G_2-structure is the asymptotic link of the asymptotically conical $Spin(7)$-manifold constructed by Bryant and Salamon [4] on the total space X of the positive spinor bundle of S^4. This bundle is $\mathcal{O}(-1)$ over $\mathbb{H}P^1$ with the orientation reversed. Since this space has $\sigma = 1$ and $\chi = 2$, it follows that $\nu(\varphi_{sq}) = 2 - 3 = -1$. (In fact, φ_{sq} is homotopic to $\bar{\varphi}_{rd}$; if we glue X and B^8 to form $\mathbb{H}P^2$ then we can interpolate to define a $Spin(7)$-structure on $\mathbb{H}P^2$.)

1.2. **The affine difference D, spinors and the ν-invariant.** An important feature of homotopy classes of G_2-structures is that the identification $\mathcal{G}_2^h(M) \equiv \mathbb{Z}$ from Lemma 1.1 should be regarded as affine. There is no preferred base point, but Lemma 1.1 has the following consequence.

Lemma 1.7. *For any pair of G_2-structures φ', φ on M there is a difference $D(\varphi, \varphi') \in \mathbb{Z}$ such that $(\mathcal{G}_2^h(M), D) \cong (\mathbb{Z}, \text{subtraction})$, i.e. $D(\varphi, \varphi') = 0$ if and only if φ is homotopic to φ', and*

$$D(\varphi, \varphi') + D(\varphi', \varphi'') = D(\varphi, \varphi''). \tag{5}$$

To understand the relationship between D and ν, we first explain the reasoning which goes into the proof of Lemma 1.1. As we describe in §2.2, a choice of Riemannian metric and unit spinor field on the spin manifold M defines a G_2-structure. Because any two Riemannian metrics are homotopic, this sets up a bijection between $\mathcal{G}_2^h(M)$ and homotopy classes of sections the unit spinor bundle. This is an S^7-bundle, and Lemma 1.1 follows from obstruction theory for sections of sphere bundles.

We can both describe D in concrete terms and prove Lemma 1.7 by counting zeros of homotopies of spinor fields (see §3.1). With this understanding of D, the following lemma is elementary.

Lemma 1.8. *Let φ, φ' be G_2-structures on M. Suppose W is a compact 8-manifold with $Spin(7)$-structure ψ such that $\partial(W, \psi) = (M, \varphi) \sqcup (-M, -\varphi')$, and let $\overline{W}$ be the closed spin 8-manifold formed by identifying the two boundary components. Then*

$$D(\varphi, \varphi') = -e_+(\overline{W}). \tag{6}$$

Combining Lemma 1.8 with the characteristic class formula (1), the mod 24 residue of $D(\varphi, \varphi')$ can be computed from just the signature and Euler characteristic of $\overline{W}$, which equal those of W. So while D only makes sense as an "affine" invariant, its mod 24 residue is related to the "absolute" invariant ν.

Proposition 1.9. *Let φ and φ' be G_2-structures on M. Then*

$$\nu(\varphi) - \nu(\varphi') \equiv -2D(\varphi, \varphi') \mod 48. \tag{7}$$

1.3. **The ν-invariant for manifolds with G_2 holonomy.** The exceptional Lie group G_2 also occurs as an exceptional case in the classification of Riemannian holonomy groups due to Berger [3]. It is immediate from the definitions that a metric on a 7-manifold M has holonomy contained in G_2 if and only if it is induced by a G_2-structure $\varphi \in \Omega^3(M)$ that is parallel. The covariant derivative $\nabla\varphi$ of φ with respect to the Levi-Civita connection ∇ of its induced metric can be identified with the intrinsic torsion of the G_2-structure, so metrics with holonomy in G_2 correspond to torsion-free G_2-structures [20, Corollary 2.2, §11].

One can define a moduli space of torsion-free G_2-structures on a fixed closed G_2-manifold M, which is locally diffeomorphic to $H^3_{dR}(M)$. But while the local structure is well understood, little is known about the global structure. One basic question is whether the moduli space is connected, *i.e.* whether any pair of torsion-free G_2-structures are equivalent up to deformation through torsion-free G_2-structures and diffeomorphism. If one could find examples of diffeomorphic G_2-manifolds where the associated G_2-structures have different values for ν, this would prove that the moduli space is disconnected.

Finding compact manifolds with holonomy G_2 is a hard problem. The known constructions solve the non-linear PDE $\nabla\varphi = 0$ using gluing methods. Joyce [12] found the first examples by desingularising flat orbifolds, and later Kovalev [15] implemented a 'twisted connected sum' construction. In [8], the classification theory of closed 2-connected 7-manifolds is used to find examples of twisted connected sum G_2-manifolds which are diffeomorphic, but without any evidence either way as to whether the G_2-structures are in the same component of the moduli space.

The twisted connected sum G_2-manifolds are constructed by gluing a pair of pieces of the form $S^1 \times V$, where V are asymptotically cylindrical Calabi-Yau 3-folds with asymptotic ends $\mathbb{R} \times S^1 \times K3$. We review this construction in §4.4 and then compute ν for all such G_2-structures.

Theorem 1.10. *If (M, φ) is a twisted connected sum then $\nu(\varphi) = 24$.*

We carry out this calculation in Section 4 by finding an explicit $Spin(7)$-bordism from a twisted connected sum G_2-structure φ to a G_2-structure that is a product of structures on lower-dimensional manifolds, for which ν is easier to evaluate.

Remark 1.11. We point out that Theorem 1.10 does not necessarily apply to more general gluings of asymptotically cylindrical G_2-manifolds.

For all the explicit examples of pairs of diffeomorphic G_2-manifolds found in [8], Corollary 1.15 implies that ν classifies the homotopy classes of G_2-structures up to diffeomorphism. So diffeomorphisms between these G_2-manifolds can always be chosen so that the corresponding torsion-free G_2-structures are homotopic. The question whether they can be connected by a path of torsion-free G_2-structures, so that they are in the same component of the moduli space of G_2-metrics, remains open.

A small number of the G_2-manifolds M constructed by Joyce [13, §12.8.4] have $\chi_{\mathbb{Q}}(M) = 1$, so those torsion-free G_2-structures have odd $\nu \neq 24$. We do not currently know how to compute ν for these G_2-manifolds.

1.4. **Counting deformation classes of G_2-structures.** We can think of the set $\mathcal{G}_2^d(M)$ of deformation-equivalence classes of G_2-structures as the quotient of $\mathcal{G}_2^h(M)$ under the action

$$\mathcal{G}_2^h(M) \times \mathrm{Diff}_{Spin}(M) \to \mathcal{G}_2^h(M), \quad ([\varphi], f) \mapsto [f^*\varphi].$$

The deformation invariance of ν implies that this action on $\mathcal{G}_2^h(M) \cong \mathbb{Z}$ is by translation by some multiple of 24, so that $\mathcal{G}_2^d(M)$ has at least 24 elements. To determine to what extent ν classifies elements of $\mathcal{G}_2^d(M)$ we need to understand precisely which multiples of 24 are realised as translations. Combining the characteristic class formula (1) with Lemma 1.8 we arrive at

Proposition 1.12. *Let $f\colon M \cong M$ be a spin diffeomorphism with mapping torus T_f. Then*

$$D(\varphi, f^*\varphi) = -24\widehat{A}(T_f) \in \mathbb{Z}.$$

The possible values of $\widehat{A}(T_f)$ are closely related to the spin characteristic class $\frac{p_1}{2}(M)$ (see §2.4). If X is a spin manifold let $p_X = \frac{p_1}{2}(X) \in H^4(X)$. We define the positive integer $d_\pi(X)$ by the equation

$$\langle p_X, H_4(X)\rangle = d_\pi(X) \cdot \mathbb{Z}.$$

In §5.1 we define $d_\infty(X)$, a more refined integer invariant of the pair $(H^4(X), p_X)$, with the property that when $H^4(X)$ is torsion-free then $d_\infty(X) = d_\pi(X)$. We remark that $d_\infty(M)$ and $d_\pi(M)$ are both even when M is a closed spin 7-manifold (see Lemma 2.6 and §5.1). For $\frac{a}{b}$ a fraction without common factors, denote $\mathrm{Num}\left(\frac{a}{b}\right) = a$.

Theorem 1.13. *Let M be a closed connected 7-dimensional spin manifold with G_2-structures φ and φ' and let $f\colon M \cong M$ be a spin diffeomorphism.*

(i) $D(\varphi, f^*\varphi) \in 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{224}\right) \cdot \mathbb{Z}$.

(ii) *If $H^4(M)$ has no 2-torsion then* $D(\varphi, f^*\varphi) \in 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{112}\right) \cdot \mathbb{Z}$.

(iii) *Suppose that there is a spin diffeomorphism $M \cong N\#M_0$ where M_0 is 2-connected and that $D(\varphi, \varphi') \in 24 \cdot \mathrm{Num}\left(\frac{d_\pi(M_0)}{112}\right) \cdot \mathbb{Z}$. Then there is a spin diffeomorphism $g\colon M \cong M$ such that φ' and $g^*\varphi$ are homotopic.*

From Theorem 1.13 we immediately deduce the following corollary concerning the size of $\mathcal{G}_2^d(M)$.

Corollary 1.14. *Let M be a closed connected 7-dimensional spin manifold.*

(i) *$\mathcal{G}_2^h(M) \equiv \mathcal{G}_2^d(M)$ if and only if $|\mathcal{G}_2^d(M)| = \infty$ if and only if $p_M = 0 \in H^4(M;\mathbb{Q})$.*

(ii) *If $p_M \neq 0 \in H^4(M;\mathbb{Q})$ then $|\mathcal{G}_2^d(M)| \leq 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{224}\right)$.*

(iii) *If $p_M \neq 0 \in H^4(M;\mathbb{Q})$ and if $H^4(M)$ has no 2-torsion then $|\mathcal{G}_2^d(M)| \leq 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{112}\right)$.*

(iv) *If M is 2-connected, $H^4(M)$ is torsion free and $p_M \neq 0$ then, because $d_\infty(M) = d_\pi(M)$, $|\mathcal{G}_2^d(M)| = 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{112}\right)$.*

The next corollary identifies a class of spin 7-manifolds M for which ν is a complete invariant of deformation classes of G_2-structures on M.

Corollary 1.15. *If $M \cong N\sharp M_0$ where M_0 is 2-connected and $d_\pi(M)$ divides 112 then we have $|\mathcal{G}_2^d(M)| = 24$. Hence two G_2-structures φ and φ' on M are deformation equivalent if and only if $\nu(\varphi) = \nu(\varphi')$.*

Determining $|\mathcal{G}_2^d(M)|$ precisely for a general spin 7-manifold M seems to be a complicated problem. As a first step towards solving this problem, we have formulated Conjecture 5.7 for the case where M is 2-connected.

1.5. **Further problems.** All twisted connected sum G_2-manifolds have $d_\pi(M)$ a divisor of 24. A number of examples with $d_\pi(M) = 12$ are exhibited in [8], and it seems likely that a more exhaustive search will provide diffeomorphic pairs of such twisted connected sums. Corollary 1.14(iv) applies to those M, so $\mathcal{G}_2^d(M)$ has $k = 72$ elements; hence they are not classified by ν (Corollary 1.15 does not apply). Is there a canonical way to define a "refinement" $\tilde{\nu} : \mathcal{G}_2^d(M) \to \mathbb{Z}_{2k}$ such that $2D(\varphi, \varphi') \equiv \tilde{\nu}(\varphi) - \tilde{\nu}(\varphi')$?

Some necessary conditions are known for a closed spin 7-manifold M to admit a metric with holonomy G_2 (see *e.g.* [13, §10.2]), but there is currently no conjecture what the right sufficient conditions would be. A refinement of this already very hard problem would be to ask: which deformation classes of G_2-structures on M contain torsion-free G_2-structures? This is of course related to the problem of whether there is any M with torsion-free G_2-structures that are not deformation-equivalent, which was one of our motivations for introducing ν. If one attempts to find torsion-free G_2-structures as limits of a flow of G_2-structures as in [27, 26], does the homotopy class of the initial G_2-structures affect the long-term behaviour?

The definition of ν in terms of a coboundary is not always amenable to explicit computations. For example the proof of Theorem 1.10 is involved, and we do not know how to evaluate ν on Joyce's orbifold resolution examples unless they are homotopic to twisted connected sums. A common theme in differential topology is to find ways to express 'extrinsic' invariants (defined in terms of a coboundary) intrinsically, *e.g.* in terms of eta invariants. One of the first invariants to be given such an analytic treatment by Atiyah, Patodi and Singer [2, Theorem 4.14] was the Adams e-invariant of framed $(4n+3)$-manifolds; in §4.3 we explain a close analogy between ν and the e-invariant in dimension 3. Can ν can be related to an eta invariant, and if so, does this relation take a simpler form for torsion-free G_2-structures?

Finally the problem of calculating the cardinality of $\mathcal{G}_2^d(M)$ remains unsolved for general M. In this direction, proving Conjecture 5.7 would determine $|\mathcal{G}_2^d(M)|$ and improve our understanding of the mapping class groups of 2-connected 7-manifolds.

Organisation. The rest of the paper is organised as follows. In Section 2 we establish preliminary results needed to define and compute ν. In Section 3 we define the affine difference $D(\varphi, \varphi')$ and the ν-invariant, establish the existence of $Spin(7)$-null bordisms for G_2-manifolds and hence prove Theorem 1.4. We also describe examples of G_2-structures on S^7 in more detail. In Section 4 we compute the ν-invariant for twisted connected sum G_2-manifolds, proving Theorem 1.10. Finally in Section 5 we describe the action of spin diffeomorphisms on $\mathcal{G}_2^h(M)$ and prove the results from §1.4.

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2. Preliminaries

In this section we describe G_2-structures and $Spin(7)$-structures on 7 and 8-manifolds, and their relationships to spinors. We also establish some basic facts about the characteristic classes of spin manifolds in dimensions 7 and 8.

2.1. **The Lie groups** $Spin(7)$ **and** G_2**.** We give a brief review of how $Spin(7)$ and G_2-structures can be characterised in terms of forms. For more detail on the differential geometry of such structures, and how they can be used in the study metrics with exceptional holonomy, see *e.g.* Salamon [20] or Joyce [13]. We defer the analogous discussion of $SU(3)$ and $SU(2)$-structures until we use it in §4.

The stabiliser in $GL(8,\mathbb{R})$ of the 4-form

$$\psi_0 = dx^{1234} + dx^{1256} + dx^{1278} + dx^{1357} - dx^{1368} - dx^{1458} - dx^{1467} - dx^{2358} - dx^{2367} - dx^{2457} + dx^{2468} + dx^{3456} + dx^{3478} + dx^{5678} \in \Lambda^4(\mathbb{R}^8)^* \tag{8}$$

is $Spin(7)$ (identified with a subgroup of $SO(8)$ by the spin representation). On an 8-dimensional manifold X, a 4-form $\psi \in \Omega^4(X)$ which is pointwise equivalent to ψ_0 defines a $Spin(7)$-structure, and induces a metric and orientation (the orientation form is ψ^2).

The exceptional Lie group G_2 can be defined as the automorphism group of $\mathbb{O}$, the normed division algebra of octonions. Equivalently, G_2 is the stabiliser in $GL(7,\mathbb{R})$ of the 3-form

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*. \tag{9}$$

On a 7-dimensional manifold M, a 3-form $\varphi \in \Omega^3(M)$ which is pointwise equivalent to φ_0 defines a G_2-structure, which induces a Riemannian metric and orientation. Note that

$$dt \wedge \varphi_0 + *\varphi_0 \cong \psi_0$$

on $\mathbb{R} \oplus \mathbb{R}^7$, so the stabiliser in $Spin(7)$ of a non-zero vector in $\mathbb{R}^8$ is exactly G_2. Therefore the product of a 7-manifold with a G_2-structure and S^1 or $\mathbb{R}$ has a natural product $Spin(7)$-structure, while if W^8 has a $Spin(7)$-structure ψ then ∂W has an induced G_2-structure given by contracting ψ with an outward pointing normal vector field; see Lemma 3.4.

Remark 2.1. If φ is G_2-structure on M^7, then $-\varphi$ is a G_2-structure too, inducing the same metric and opposite orientation (because φ_0 is equivalent to $-\varphi_0$ under the orientation-reversing diffeomorphism $-1 \in O(7)$). As warned in Example 1.5, this has the potential to cause some confusion. The product $Spin(7)$-structure $dt \wedge \varphi + *\varphi$ on $M \times [0,1]$ induces φ on the boundary component $M \times \{1\} \cong M$, and $-\varphi$ on $M \times \{0\} \cong -M$.

2.2. G_2**-structures and spinors.** In this paper we are concerned with G_2-structures on a manifold M^7 up to homotopy. Since there is an obvious way to reverse the orientation of a G_2-structure, while any two Riemannian metrics are homotopic, we may as well consider G_2-structures compatible with a fixed orientation and metric. Because G_2 is simply-connected, the inclusion $G_2 \hookrightarrow SO(7)$ lifts to $G_2 \hookrightarrow Spin(7)$. Therefore a G_2-structure on M also induces a spin structure, and we focus on studying G_2-structures compatible also with a fixed spin structure. As in the introduction, we let $\mathcal{G}_2^h(M)$ denote the homotopy classes of G_2-structures on M with a choice of spin structure.

As we already saw, G_2 is exactly the stabiliser of a non-zero vector in the spin representation S of $Spin(7)$; as a representation of G_2, S splits as the sum of a 1-dimensional trivial part and the standard 7-dimensional representation. $Spin(7)$ acts transitively on the unit sphere in S with stabiliser G_2, so $Spin(7)/G_2 \cong S^7$.

From the above, we deduce that given a spin structure on M, a compatible G_2-structure φ induces an isomorphism $S \cong \underline{\mathbb{R}} \oplus TM$ for the spinor bundle: here $\underline{\mathbb{R}}$ denotes the trivial line bundle. Hence we can associate to φ a unit section of S, well-defined up to sign. Conversely, any unit section of S defines a compatible G_2-structure. A transverse section ϕ of the spinor bundle S of a spin 7-manifold has no zeros, so defines a G_2-structure; thus a 7-manifold admits G_2-structures if and only if it is spin.

Note that ϕ and $-\phi$ are always homotopic, because they correspond to sections of the trivial part in a splitting $S \cong \underline{\mathbb{R}} \oplus TM$ and the Euler class of an oriented 7-manifold vanishes. It follows that S contains a trivial 2-plane field $K \supset \underline{\mathbb{R}}$ which accommodates a homotopy from ϕ to $-\phi$. Therefore $\mathcal{G}_2^h(M)$ can be identified with homotopy classes of unit sections of the spinor bundle. As stated in the introduction, Lemma 1.1 now follows by a standard application of obstruction theory, but we will describe the bijection $\mathcal{G}_2^h(M) \cong \mathbb{Z}$ in elementary terms in §3.1.

Remark 2.2. Let us make some further comments on the signs of the spinors. Given a principal $Spin(7)$ lift $\tilde{F}$ of the frame bundle F of M, the principal G_2-subbundles of $\tilde{F}$ are in 1-to-1 correspondence with sections of the associated unit spinor bundle. The G_2-subbundles corresponding to spinors ϕ and $-\phi$ have the same image in F, hence they define the same G_2-structure on M (they have the same 3-form φ).

While $SO(7)$ does not itself act on S, the action of $Spin(7)$ on $(S-\{0\})/\mathbb{R}^* \cong \mathbb{R}P^7$ does descend to an action of $SO(7)$. Therefore the orbit $SO(7)\varphi_0$, the set of G_2-structures on $\mathbb{R}^7$ defining the same orientation and metric as φ_0, is $SO(7)/G_2 \cong \mathbb{R}P^7$. G_2-structures compatible with a fixed orientation and metric on M but without any constraint on the spin structure therefore correspond to sections of an $\mathbb{R}P^7$ bundle. If M is not spin then this bundle has no sections. Given a spin structure, the unit sphere bundle in the associated spinor bundle is an S^7 lift of the $\mathbb{R}P^7$-bundle, and two G_2-structures induce the same spin structure if they can both be lifted to the same S^7 bundle.

2.3. ***Spin*(7)-structures and characteristic classes of *Spin*(8)-bundles.** The inclusion homomorphism $Spin(7) \hookrightarrow SO(8)$ has a lift $Spin(7) \hookrightarrow Spin(8)$. The restriction of the positive spin representation S_+ of $Spin(8)$ to $Spin(7)$ is a sum of a trivial rank 1 part and the standard 7-dimensional representation (factoring through $Spin(7) \to SO(7)$). Therefore $Spin(7) \subset Spin(8)$ can be characterised as the stabiliser of a non-zero positive spinor, and there is an obvious obstruction to the existence of $Spin(7)$-structures on an 8-manifold X: it must be spin, and because the $Spin(7)$-structure corresponds to a non-vanishing positive spinor (modulo an overall sign) the Euler class in $H^8(X)$ of the positive half-spinor bundle on X must vanish.

Let us describe briefly our conventions for orientations on the half-spin representations of $Spin(8)$. For each fixed non-zero $v \in \mathbb{R}^8$, the Clifford multiplication $\mathbb{R}^8 \times S_\pm \to S_\mp$ defines orientation-preserving isomorphisms $c_v^\pm : S_\pm \to S_\mp$. A feature of the 'triality' in dimension 8 is that the map $s_{\phi_\pm} : \mathbb{R}^8 \to S_\mp$ induced by Clifford multiplication with a fixed non-zero spinor $\phi_\pm \in S_\pm$ is an isomorphism too. The Clifford relations imply that, for $\phi_+ = v\phi_-$,

$$c_v^+ \circ s_{\phi_-} = s_{\phi_+} \circ R_v \;:\; \mathbb{R}^8 \to S_-,$$

where $R_v : \mathbb{R}^8 \to \mathbb{R}^8$ is reflection in the hyperplane orthogonal to v. Thus $s_{\phi_\pm}$ have opposite orientability. Our convention is that s_{ϕ_-} is orientation-preserving, while s_{ϕ_+} is not.

More explicitly, $\mathbb{R}^8$, S_+ and S_- can each be identified with the octonions $\mathbb{O}$ in such a way that the Clifford multiplication $\mathbb{R}^8 \times S_- \to S_+$ corresponds to the octonionic multiplication $(x,y) \mapsto xy$. Then, to satisfy the Clifford relations, $\mathbb{R}^8 \times S_+ \to S_-$ must correspond to $(x,y) \mapsto -\bar{x}y$, where $\bar{x}$ is the octonion conjugate of x. This map is orientation-reversing on the first factor.

Let X be a spin 8-manifold, $e \in H^8(X;\mathbb{Z})$ the Euler class of TX, and $e_\pm \in H^8(X;\mathbb{Z})$ the Euler classes of the half-spinor bundles $S_\pm$. More generally, for any principal $Spin(8)$-bundle on any X, let $e, e_\pm$ denote the Euler classes of the associated vector bundles of the vector and half-spin representations of $Spin(8)$. With our orientation conventions, the non-degeneracy of the Clifford product implies

$$e_+ = e + e_-. \tag{10}$$

Proposition 2.3 ([11, (3.8)])**.** *For any principal* $Spin(8)$*-bundle*

$$e_\pm = \frac{1}{16}\left(p_1^2 - 4p_2 \pm 8e\right).$$

Corollary 2.4. *If X is a closed spin 8-manifold then*

$$e_\pm(X) = 24\widehat{A}(X) + \frac{\pm\chi(X) - 3\sigma(X)}{2}.$$

Proof. Combine the definition of the $\widehat{A}$ genus

$$45 \cdot 2^7 \widehat{A}(X) = \int_X 7p_1(X)^2 - 4p_2(X)$$

and the Hirzebruch signature formula

$$45\sigma(X) = \int_X 7p_2(X) - p_1(X)^2. \qquad \square$$

Remark 2.5. Modulo torsion, the group of integral characteristic classes of a principal $Spin(8)$-bundle in dimension 8 is generated by p_1^2, p_2 and e, so we could prove Corollary 2.4 (and hence Proposition 2.3) by checking that the formula holds for the following spin 8-manifolds.

- S^8: $\chi = 2$, $\widehat{A} = \sigma = 0$, $e_\pm = \pm 1$.
- $K3 \times K3$: $\chi = 24^2$, $\sigma = (-16)^2$. $\widehat{A} = 4$ because the holonomy is $SU(2) \times SU(2)$. Because this also defines a $Spin(7)$-structure (*cf.* (13)), $e_+ = 0$ and $e_- = -\chi$.
- $\mathbb{H}P^2$: $\chi = 3$, $\sigma = 1$. $\widehat{A} = 0$ by the Lichnerowicz formula since there is a metric with positive scalar curvature. $e_- = -\chi$ because $S_- \cong -TX$ for any spin 8-manifold X with $Sp(2)Sp(1)$-structure. This structure also splits S_+ into a sum of a rank 5 and a rank 3 part, both of which have Euler class 0 as $H^3(\mathbb{H}P^2) = 0$, so $e_+ = 0$. (Alternatively, we can identify a quaternionic line subbundle of $T\mathbb{H}P^2$, like that spanned by the projection of the vector field $(q_1, q_2, q_3) \mapsto (0, q_1, q_2)$ on $\mathbb{H}^3$, with a non-vanishing section of the rank 5 part of S_+.)

2.4. **The spin characteristic class $\frac{p_1}{2}$.** Recall that the classifying space $BSpin$ is 3-connected and $\pi_4(BSpin) \cong \mathbb{Z}$. It follows that $H^4(BSpin) \cong \mathbb{Z}$ is infinite cyclic. A generator is denoted $\pm\frac{p_1}{2}$ and the notation is justified since for the canonical map $\pi\colon BSO \to BSpin$ we have $2\pi^*\frac{p_1}{2} = p_1$ where p_1 is the first Pontrjagin class. Given a spin manifold X we write

$$p_X := \frac{p_1}{2}(X) \in H^4(X).$$

The following lemma is well known but we include a proof for the reader's convenience.

Lemma 2.6. *For a closed spin 7-manifold M, $p_M \in 2H^4(M)$.*

Proof. From the definition it is clear that the mod 2 reduction of $\frac{p_1}{2}$ is w_4, the $4th$ Stiefel-Whitney class. But by Wu's formula, see *e.g.* [19, Theorem 11.14] $w_4 = v_4$ on the spin manifold M since the first three Wu classes of a spin manifold vanish. Finally $v_4(M) = 0$ since M is 7-dimensional, the Wu class satisfies $v_4 \cup x = Sq^4(x)$ for all $x \in H^3(M; \mathbb{Z}_2)$ and Sq^4 vanishes on three dimensional classes. □

3. The ν-invariant

In this section we study the set $\mathcal{G}_2^h(M)$ of homotopy classes of G_2-structures on a closed spin 7-manifold M, and prove the basic properties of the invariants D and ν. We conclude the section with some concrete examples.

3.1. **The affine difference.** Let M be a closed connected spin 7-manifold, and φ, φ' a pair of G_2-structures on M. We describe how to define the difference $D(\varphi, \varphi') \in \mathbb{Z}$ from Lemma 1.7.

A homotopy of G_2-structures is equivalent to a path of non-vanishing spinor fields. Any path of spinor fields on M can be identified with a positive spinor field ϕ on $M \times [0,1]$. We can always find ϕ with transverse zeros, such that the restriction to the two boundary components are the non-vanishing spinor fields corresponding to φ and $-\varphi'$. The intersection number $n_+(M \times [0,1], \varphi, \varphi')$ of ϕ with the zero section is independent of ϕ, and we take this as the definition of $D(\varphi, \varphi')$.

It is obvious from this definition that the affine relation (5) holds. If $n_+(M \times [0,1], \varphi, \varphi') = 0$ then ϕ can be chosen to be non-vanishing, so φ and φ' are homotopic if and only if $D(\varphi, \varphi') = 0$. Given φ we can construct φ' such that $D(\varphi, \varphi') = 1$ by modifying the defining spinor of φ in a 7-disc B^7: in a local trivialisation we change it from a constant map $B^7 \to S^7$ to a degree 1 map. Thus D can take any integer value, so D really corresponds to the difference function under a bijection $\mathbb{Z} \cong \mathcal{G}_2^h(M)$, completing the proof of Lemma 1.7.

To compute $D(\varphi, \varphi')$, we can consider more general spin 8-manifolds W with boundary $M \sqcup -M$. Generalising the above, let $n_+(W, \varphi, \varphi')$ be the intersection number with the zero section of a positive spinor whose restriction to the two boundary components correspond to φ and $-\varphi'$. Gluing the boundary components of W gives a closed spin 8-manifold $\overline{W}$. Clearly $\overline{W}$ has a positive spinor field whose intersection number with the zero section is $n_+(W, \varphi, \varphi') - D(\varphi, \varphi')$. Hence we can compute D as

$$D(\varphi, \varphi') = n_+(W, \varphi, \varphi') - e_+(\overline{W}). \tag{11}$$

3.2. **The definition of ν.** Let M be a closed spin 7-manifold with G_2-structure φ, and W a compact spin 8-manifold with $\partial W = M$. Such W always exist since the bordism group Ω_7^{Spin} is trivial. The restriction of the half-spinor bundles $S_\pm$ of W to M are isomorphic to the spinor bundle on M. The composition $S_{+|M} \to S_{-|M}$ of these isomorphisms is Clifford multiplication by a unit normal vector field to the boundary. Let $n_\pm(W, \varphi)$ be the intersection number with the zero section of a section of $S_\pm$ whose restriction to M is the non-vanishing spinor field defining φ. Let

$$\bar{\nu}(W, \varphi) := -2n_+(W, \varphi) + \chi(W) - 3\sigma(W) \in \mathbb{Z}.$$

Reversing the orientations, $-W$ is a spin 8-manifold whose boundary $-M$ is equipped with a G_2-structure $-\varphi$.

Lemma 3.1. *Let W be a compact spin 8-manifold, and φ a G_2-structure on $M = \partial W$.*

(i) $\bar{\nu}(W, \varphi) \equiv \chi_{\mathbb{Q}}(M) \mod 2$
(ii) $\bar{\nu}(-W, -\varphi) = -\bar{\nu}(W, \varphi)$
(iii) *Let φ' be another G_2-structure on M. If W' is another spin 8-manifold with $\partial W' = M$ then the closed spin 8-manifold $X = W \cup_{\mathrm{Id}_M} (-W')$ has*

$$-48\widehat{A}(X) = \bar{\nu}(W, \varphi) - \bar{\nu}(W', \varphi') + 2D(\varphi, \varphi').$$

Proof. (i) For W^{4n} any compact oriented manifold with boundary, $\sigma(W)$ is by definition the signature of a non-degenerate symmetric form on the image $H_0^{2n}(W)$ of $H_c^{2n}(W) \to H^{2n}(W)$. The exact sequence $0 \to H_c^0(W) \to H^0(W) \to \cdots \to H^{2n-1}(\partial W) \to H_c^{2n}(W) \to H_0^{2n}(W) \to 0$ shows

$$\sigma(W)+\chi(W) \equiv \dim H_0^{2n}(W)+b^{2n}(W)+\sum_{i=0}^{2n-1} b^i(W)+b^{4n-i}(W) \equiv \sum_{i=0}^{2n-1} b^i(\partial W) \equiv \chi_{\mathbb{Q}}(\partial W) \mod 2.$$

(ii) Let V be a vector field on W that is a unit outward-pointing normal field along M, and $\phi \in \Gamma(S_+)$ a spinor field whose restriction to M induces φ. Then the restriction of the Clifford product $V \cdot \phi \in \Gamma(S_-)$ also induces φ. By the Poincare-Hopf index theorem, the number of zeros of V is $\chi(M)$, so $n_-(W, \varphi) = n_+(W, \varphi) - \chi(W)$ (these signs are compatible with (10)).

Reversing the orientations swaps sections of S_+ and S_-, and reverses the signs assigned to the zeros, so $n_+(-W, -\varphi) = -n_-(W, \varphi)$. It also reverses the signature, but preserves the Euler characteristic. Thus

$$\bar{\nu}(-W, -\varphi) = 2n_-(W, \varphi) + \chi(W) + 3\sigma(W) = 2n_+(W, \varphi) - 2\chi(W) + \chi(W) + 3\sigma(W) = -\bar{\nu}(W, \varphi).$$

(iii) $\sigma(W)+\sigma(-W') = \sigma(X)$ by Novikov additivity. $\chi(W)+\chi(-W') = \chi(X)$ because $\chi(M) = 0$. $W \sqcup (-W')$ is a bordism of M to itself with $n_+(W \sqcup (-W'), \varphi, \varphi') = n_+(W, \varphi) + n_+(-W', -\varphi')$. Hence by Corollary 2.4 and (11)

$$\begin{aligned}\bar{\nu}(W, \varphi) - \bar{\nu}(W', \varphi') &= -2(n_+(W, \varphi)+n_+(-W', -\varphi'))+\chi(W)+\chi(-W')-3(\sigma(W)+\sigma(-W')) = \\ &- 2n_+(W \sqcup (-W'), \varphi, \varphi') + 2e_+(X) - 48\widehat{A}(X) = -2D(\varphi, \varphi') - 48\widehat{A}(X). \quad \square\end{aligned}$$

Corollary 3.2. $\nu(\varphi) := \bar{\nu}(W, \varphi) \mod 48 \in \mathbb{Z}_{48}$ *is independent of the choice of W, and*

$$\nu(\varphi) - \nu(\varphi') \equiv -2D(\varphi, \varphi') \mod 48.$$

This essentially proves Theorem 1.4 and Proposition 1.9. To complete the proofs it remains only to show the existence of *Spin*(7)-coboundaries, since Definition 1.2 is phrased in terms of those. We show the existence of the require *Spin*(7)-coboundaries in the following subsection.

3.3. ***Spin*(7)-bordisms.** Let φ, φ' be G_2-structures on closed 7-manifolds M, M'. A *Spin*(7)-bordism from (M, φ) to (M', φ') is a compact 8-manifold with boundary $M \sqcup -M'$ and a *Spin*(7)-structure ψ that restricts to φ and $-\varphi'$ on M and $-M'$. Clearly, there is a topologically trivial *Spin*(7)-bordism W (*i.e.* there is a diffeomorphism $W \cong M \times [0, 1]$, but it does not have to preserve the *Spin*(7)-structure) from φ to φ' if and only if they are deformation-equivalent, *i.e.* $f^*\varphi'$ is homotopic to φ for some diffeomorphism f.

Remark 3.3. If W is a $Spin(7)$-bordism from (M, φ) to (M', φ') then t is also a $Spin(7)$-bordism from $(-M', -\varphi')$ to $(-M, -\varphi)$. However, it does not follow in general that $-W$ has a $Spin(7)$-structure making it a $Spin(7)$-bordism from (M', φ') to (M, φ) (because the orientation of a $Spin(7)$-structure cannot be reversed). In particular, if W is a $Spin(7)$-coboundary for (M, φ) then $-W$ is not necessarily a $Spin(7)$-coboundary for $(-M, -\varphi)$, unless $\chi(W) = 0$, *cf.* proof of Lemma 3.1(ii).

A $Spin(7)$-structure ψ induces a non-vanishing positive spinor field ϕ on W, so $n_+(W, \varphi, \varphi') = 0$. In particular, when φ and φ' are G_2-structures on the same manifold $M = M'$, Lemma 1.8 follows from (11). Similarly, if W is a $Spin(7)$-coboundary for (M, φ) then $\bar{\nu}(W, \varphi) = \chi(W) - 3\sigma(W)$, so Corollary 3.2 together with Lemma 3.4(ii) imply Theorem 1.4.

Lemma 3.4.

(i) *For a connected compact spin 8-manifold W with connected boundary M, there is a unique homotopy class of G_2-structures on M that bound $Spin(7)$-structures on W.*

(ii) *Any two G_2-structures are $Spin(7)$-bordant. In particular, any G_2-structure has a $Spin(7)$ coboundary.*

Proof. If W is connected with non-empty boundary then there is no obstruction to defining a non-vanishing positive spinor field on W, so there is some G_2-structure φ on M that bounds a $Spin(7)$-structure on W. If φ' is another G_2-structure bounding a $Spin(7)$-structure on W, consider an arbitrary spin filling W' of $-M$, and let $-\varphi''$ be a G_2-structure on $-M$ that bounds a $Spin(7)$-structure on W'. Then $W \sqcup W'$ admits two $Spin(7)$-structures that define bordisms from φ and φ', respectively, to φ''. Hence

$$D(\varphi, \varphi') = D(\varphi, \varphi'') - D(\varphi', \varphi'') = 0,$$

and φ and φ' must be homotopic.

For (ii), take any spin filling W of M, and let φ be a G_2-structure on M that bounds a $Spin(7)$-structure. In order to find a $Spin(7)$-coboundary for some other φ' with $D(\varphi, \varphi') = \pm k$, we use that if X and X' are closed spin 8-manifolds then (since $\widehat{A}$ and σ are bordism-invariants, and in particular additive under connected sums) Corollary 2.4 implies that

$$e_+(X \# X') = e_+(X) + e_+(X') - 1.$$

(We could also see that for any pair of positive spinor fields ϕ, ϕ' on X, X' one can define a spinor field on $X \# X'$ that equals ϕ and ϕ' outside the connecting neck, and with a single zero on the neck.) Therefore φ' will bound a $Spin(7)$-structure on W' the connected sum of W with k copies of a manifold with $e_+ = 2$ or 0, *e.g.* $S^4 \times S^4$ or T^8. □

3.4. **Examples of G_2-structures on S^7.** To make the discussion more concrete, we elaborate on Examples 1.5 and 1.6 from the introduction.

Example 3.5. The standard round G_2-structure φ_{rd} on S^7 is given by contracting the constant 4-form ψ_0 on $\mathbb{R}^8$ with the outward normal unit vector field. Then trivially (B^8, ψ_0) is a $Spin(7)$-coboundary for (S^7, φ_{rd}). The contraction of ψ_0 with the unit inward normal of S^7 gives $-\varphi_{rd}$; this is still a G_2-structure, but compatible with the opposite orientation of S^7. If $r : \mathbb{R}^8 \to \mathbb{R}^8$ is an (orientation-reversing) reflection, then $\bar{\varphi}_{rd} = r^*(-\varphi_{rd})$ is a G_2-structure inducing the same orientation as φ_{rd}. $W = (B^8, \psi_0) \sqcup (-B^8, r^*\psi_0)$ has boundary $(S^7, \varphi_{rd}) \sqcup (-S^7, r^*\varphi_{rd})$, so gives a $Spin(7)$-bordism from φ_{rd} to $\bar{\varphi}_{rd}$. In this case $\overline{W} = S^8$, so $D(\varphi_{rd}, \bar{\varphi}_{rd}) = -e_+(S^8) = -1$.

For G_2-structures on S^7, D can also be described more directly. The spinor bundle of S^7 can be trivialised by identifying it with the restriction of the positive half-spinor bundle on B^8, thus up to homotopy, G_2-structures on S^7 can be identified with maps from S^7 to the unit sphere in S_+. The trivialisation is equivariant under the natural actions of $Spin(8)$ on S^7 (via $SO(8)$) and on its positive spin representation. The difference D between two G_2-structures on S^7 equals the difference of the degrees of the corresponding maps $S^7 \to S^7$. This is not necessarily easier than using (11), but it is reassuring when it gives the same answer.

Example 3.6. By definition, the standard round G_2-structure φ_{rd} corresponds to a constant map $f_{rd} : x \mapsto \phi_0$. The G_2-structure φ_{rd} is invariant under the action of $Spin(7)$, and so is f_{rd}, in the sense that $f_{rd}(gx) = \phi_0 = g\phi_0 = gf_{rd}(x)$ for any $g \in Spin(7)$.

Let r be a reflection of S^7, and $\bar{\varphi}_{rd} = r^*(-\varphi_{rd})$ as above. Then $\bar{\varphi}_{rd}$ is invariant under the action of the conjugate subgroup $rSpin(7)r \subset Spin(8)$. If $x_0 \in S^7$ is a vector orthogonal to the hyperplane of the reflection, then φ_{rd} and $\bar{\varphi}_{rd}$ take the same value at x_0. Thus $\bar{f}_{rd}(x_0) = \phi_0$, and $\bar{f}_{rd}(rgrx_0) = (rgr)\phi_0$ for any $g \in Spin(7)$. The outer automorphism on $Spin(8)$ of conjugating by a reflection swaps the positive and negative spin representations, so the representation of $Spin(7)$ on S_+ by $(g, \phi) \mapsto (rgr)\phi$ is isomorphic to S_-, which as a representation of $Spin(7)$ is in turn isomorphic to the vector representation. Hence $\bar{f}_{rd}$ is a bijection $S^7 \to S^7$. While we have not kept track of the orientations, this agrees with the claim in Example 3.5 that $D(\varphi_{rd}, \bar{\varphi}_{rd}) = -1$.

Example 3.7. There is an orientation-reversing diffeomorphism q from the unit ball subbundle of $\mathcal{O}(-1)$ on $\mathbb{H}P^1$ (whose boundary is naturally S^7) to the Bryant-Salamon asymptotically conical $Spin(7)$-manifold X, such that the pull-back of the $Spin(7)$-structure is invariant under the natural $Sp(2)Sp(1)$ action. Let (Σ, φ_{sq}) be the link of the cone of X, with its squashed nearly parallel G_2-structure. Then $-q^*\varphi_{sq}$ is a $Sp(2)Sp(1)$-invariant G_2-structure on S^7, compatible with the standard orientation. The associated map $\bar{f}_{sq} : S^7 \to S^7$ is $Sp(2)Sp(1)$-equivariant. Because $Sp(2)Sp(1)$ does not act transitively on the unit sphere in S_+, $\bar{f}_{sq}$ has degree 0. Hence $-q^*\varphi_{sq}$ is homotopic to φ_{rd}. Equivalently, if we compose with a reflection r of the sphere to get an orientation-preserving diffeomorphism $p = qr : S^7 \to \Sigma$, then $p^*\varphi_{sq}$ is homotopic to $\bar{\varphi}_{rd}$.

Example 3.8. The G_2-structure φ_{rd} is invariant under the order 4 diffeomorphism given by multiplication by i on $\mathbb{C}^4$, so descends to a G_2-structure $\varphi_{rd}/\mathbb{Z}_4$ on the quotient $S^7/\mathbb{Z}_4$. This is the boundary of the unit disc bundle of $\mathcal{O}(-4)$ on $\mathbb{C}P^3$ (the canonical bundle of $\mathbb{C}P^3$), which has an $SU(4)$-structure restricting to $\varphi_{rd}/\mathbb{Z}_4$ (indeed, the total space admits a Calabi-Yau metric asymptotic to $\mathbb{C}^4/\mathbb{Z}_4$, *cf.* Calabi [6, §4]). The self-intersection number of a hyperplane in the zero-section is -4, so $\sigma = -1$, and $\nu(\varphi_{rd}/\mathbb{Z}_4) = 4 + 3 = 7$.

Remark 3.9. If φ and φ' are G_2-structures on the same closed spin 7-manifold M and $\pi : \tilde{M} \to M$ is a degree k covering map, then $D(\pi^*\varphi, \pi^*\varphi') = kD(\varphi, \varphi')$. However, the fact that φ_{rd} and $\bar{\varphi}_{rd}$ are both invariant under the antipodal map on S^7 is not incompatible with $D(\varphi_{rd}, \bar{\varphi}_{rd})$ being odd: the G_2-structures they define on $\mathbb{R}P^7 = S^7/\pm 1$ induce different spin structures.

4. ν of twisted connected sum G_2-manifolds

The purpose of introducing the invariant ν is to give a tool for studying the homotopy classes of G_2-structures. For this tool to be of any value, we also need to be able to compute ν in practice. We now show how the definition of ν in terms of $Spin(7)$-bordisms allows us to compute it for the large class of 'twisted connected sum' manifolds with holonomy G_2. Before describing the twisted connected sums, we explain how to compute ν of G_2-structures defined as products of structures on lower-dimensional manifolds. This is then used in the proof of Theorem 1.10, that the torsion-free G_2-structures of twisted connected sum G_2-manifolds always have $\nu = 24$. In the process we describe how an analogue of the ν invariant in dimension 3 is related to the Adams e-invariant classifying the bordism group of framed 3-manifolds.

4.1. $SU(3)$ and $SU(2)$-structures.

Let us first describe $SU(3)$ and $SU(2)$-structures in terms of forms, along the lines of §2.1.

Let $z^k = x^{2k-1} + ix^{2k}$ be complex coordinates on $\mathbb{R}^6$. Then the stabiliser in $GL(6, \mathbb{R})$ of the pair of forms

$$\Omega_0 = dz^1 \wedge dz^2 \wedge dz^3 \in \Lambda^3(\mathbb{R}^6)^* \otimes \mathbb{C}$$
$$\omega_0 = \tfrac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3) \in \Lambda^2(\mathbb{R}^6)^*$$

is $SU(3)$. An $SU(3)$-structure (Ω, ω) on a 6-manifold induces a Riemannian metric, almost complex structure and orientation (the volume form is $-\frac{i}{8}\Omega \wedge \bar{\Omega} = \frac{1}{6}\omega^3$). On $\mathbb{R} \oplus \mathbb{R}^6$

$$dt \wedge \omega_0 + \operatorname{Re}\Omega_0 \cong \varphi_0, \tag{12}$$

and $SU(3)$ is exactly the stabiliser in G_2 of a non-zero vector in $\mathbb{R}^7$. The product of a 6-manifold with $SU(3)$-structure and S^1 or $\mathbb{R}$ has a product G_2-structure, while the boundary of a 7-manifold with G_2-structure has an induced $SU(3)$-structure.

The stabiliser in $GL(4,\mathbb{R})$ of the triple of forms

$$\omega_0^I = dx^{12} + dx^{34},\ \omega_0^J = dx^{13} - dx^{24},\ \omega_0^K = dx^{14} + dx^{23} \in \Lambda^2(\mathbb{R}^4)^*$$

is $SU(2)$. The stabiliser in $SU(2)$ of a non-zero vector is clearly trivial, and the boundary of a 4-manifold W with $SU(2)$-structure $(\omega^I, \omega^J, \omega^K)$ has a natural coframe defined by contracting each of the three 2-forms with an outward pointing normal vector field.

If e^1, e^2, e^3 is a coframe on $\mathbb{R}^3$ then

$$e^{123} + e^1 \wedge \omega_0^I + e^2 \wedge \omega_0^J + e^3 \wedge \omega_0^K \cong \varphi_0$$

on $\mathbb{R}^3 \oplus \mathbb{R}^4$. Therefore the product of a parallelised 3-manifold and a 4-manifold with $SU(2)$-structure has a natural product G_2-structure. Similarly, if we let $\omega_1^I, \omega_1^J, \omega_1^K$ denote an equivalent triple of 2-forms on a second copy of $\mathbb{R}^4$, and $\mathrm{vol}_0 = \frac{1}{2}(\omega_0^I)^2$ etc, then

$$\mathrm{vol}_0 + \omega_0^I \wedge \omega_1^I + \omega_0^J \wedge \omega_1^J + \omega_0^K \wedge \omega_1^K + \mathrm{vol}_1 \cong \psi_0 \tag{13}$$

on $\mathbb{R}^4 \oplus \mathbb{R}^4$, so the product of two 4-manifolds with $SU(2)$-structures has a natural product $Spin(7)$-structure.

4.2. **Product G_2-structures and spinors.** Above we described two types of product G_2-structures. In order to compute ν of such products, we need to describe $SU(3)$ and $SU(2)$ in terms of spinors.

The half-spin representations $S_\pm$ of $Spin(6) \cong SU(4)$ are the standard 4-dimensional representation of $SU(4)$ and its dual. The inclusion $SU(3) \hookrightarrow SO(6)$ lifts to the obvious inclusion $SU(3) \hookrightarrow SU(4)$, so the stabiliser of a non-zero element in S_+ is exactly $SU(3)$. Hence, analogously to §2.2, $SU(3)$-structures on a 6-manifold Y compatible with a fixed spin structure and metric can be defined by positive unit spinor fields. It is immediately clear that any two are homotopic.

If Y is the boundary of a spin 7-manifold M, then the half-spinor bundles on Y are both isomorphic, as real vector bundles, to the restriction of the spinor bundle from M. As there is no obstruction to extending a non-vanishing section of a rank 8 bundle on M from the boundary to the interior, it follows that any $SU(3)$-structure on Y is induced as the boundary of a G_2-structure on M.

Lemma 4.1. *If Y is a 6-manifold with an $SU(3)$-structure (Ω, ω), then the product G_2-structure $\varphi = d\theta \wedge \omega + \mathrm{Re}\,\Omega$ on $S^1 \times Y$ has $\nu(\varphi) = 0$.*

Proof. Any spin 6-manifold Y bounds some spin 7-manifold M, as the bordism group Ω_6^{Spin} is trivial. Then any product G_2-structure φ on $S^1 \times Y$ bounds a product $Spin(7)$-structure on $S^1 \times M$. The S^1 factor makes $\sigma(S^1 \times M) = \chi(S^1 \times M) = 0$, so $\nu(\varphi) = 0$. □

Now we look at spinors in dimension 3 and 4. The spin representations of $Spin(4) \cong SU(2) \times SU(2)$ are the standard 2-dimensional complex representations of the two factors. Therefore the stabiliser of a non-zero positive spinor is one of the $SU(2)$ factors, and a non-zero spinor field on a spin 4-manifold defines an $SU(2)$-structure.

The spin representation of $Spin(3) \cong SU(2)$ is again the standard representation of $SU(2)$. The stabiliser of a non-zero spinor is trivial, so a non-zero spinor field defines a parallelism, *i.e.* a trivialisation of the tangent bundle. If X is a 4-manifold with boundary M, then the restriction of either the positive or negative spinor bundle to M is isomorphic to the spinor bundle of M. The analogue in dimension 4 of Corollary 2.4 is that

$$e_\pm(X) = \tfrac{3}{4}\sigma(X) \pm \tfrac{1}{2}\chi(X) \tag{14}$$

for any closed spin 4-manifold X. (It suffices to check on S^4 and $K3$.)

For a closed spin 3-manifold M, the spin structure induces a homogeneous quadratic refinement of the $\mathbb{Q}/\mathbb{Z}$-valued torsion linking form b on $TH^2(M)$. This is a function $q\colon TH^2(M) \to \mathbb{Q}/2\mathbb{Z}$ such that $q(x+y) = q(x) + q(y) + 2b(x,y)$ and $q(-x) = q(x)$. The function q may be defined as follows.

The spin manifold M bounds a spin 4-manifold W since the bordism group Ω_3^{Spin} vanishes. The intersection form of W, $(\lambda_W, H_2(W)/\mathrm{Tors})$, is an even symmetric bilinear form which splits as

$$(\lambda_W, H_2(W)/\mathrm{Tors}) \cong (\lambda, H) \oplus (0, R)$$

where (λ, H) is non-degenerate. If $H^\# \subset H \otimes \mathbb{Q}$ is the dual lattice of (λ, H), consisting of those elements in $H \otimes \mathbb{Q}$ which pair integrally with H, then $TH^2(M) \cong H^\#/H$ and q is defined as the function

$$q\colon H^\#/H \to \mathbb{Q}/\mathbb{Z}, \quad [u] \mapsto \frac{1}{2}\lambda(u,u).$$

Define the Gauss sum $GS(M)$ by

$$GS(M) = \frac{1}{2\pi} \arg \sum_{x \in TH^2(M)} e^{2\pi i q(x)}.$$

Then by the theorem of Milgram in [18, Appendix 4], $GS(M) \in \mathbb{Z}_8$ and $\sigma(W) = GS(M) \mod 8$ for any spin coboundary W of M.

Lemma 4.2. *Let X be a 4-manifold with an $SU(2)$-structure $(\omega^I, \omega^J, \omega^K)$ and M a 3-fold with a coframe field (e^1, e^2, e^3). Then the product G_2-structure $\varphi = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge \omega^I + e^2 \wedge \omega^J + e^3 \wedge \omega^K$ on $M \times X$ has $\nu(\varphi) = 12\widehat{A}(X)(\chi_{\mathbb{Q}}(M) + GS(M)) \mod 48$.*

Proof. Pick a spin coboundary W of M. Let $n_+(W,t)$ be the intersection number with the zero section of a positive spinor field on W whose restriction to M is the defining spinor field of the parallelism t equivalent to the coframe field. We can apply connected sums with T^4 or $S^2 \times S^2$ to make $n_+(W,t) = 0$ (this is the same argument as in Lemma 3.4). So we can assume that W has an $SU(2)$-structure.

Since $\sigma(W) = GS(M) \mod 8$ and $\sigma(W) + \chi(W) = \chi_{\mathbb{Q}}(M) \mod 2$, we find that

$$\chi(W) = \chi_{\mathbb{Q}}(M) + GS(M) \mod 2. \tag{15}$$

On the other hand, $\widehat{A}(X)$ is an even integer for any closed spin 4-manifold, and $\sigma(X) = -8\widehat{A}(X)$. If X has an $SU(2)$-structure, then $e_+(X) = 0$, so (14) implies $\chi(X) = 12\widehat{A}(X)$. $W \times X$ is a $Spin(7)$-coboundary for φ, so

$$\begin{aligned}\nu(\varphi) = \chi(W \times X) - 3\sigma(W \times X) &= 12\widehat{A}(X)\chi(W) - 3\sigma(W)(-16\widehat{A}(X)) \\ &= 12\widehat{A}(X)(\chi_{\mathbb{Q}}(M) + GS(M)) \mod 48. \quad \square\end{aligned}$$

4.3. **Framings of 3-manifolds.** There is a relation between G_2-structures and 3-dimensional geometry in that both involve cross products. For a spin 3-manifold M, we noted above that a non-vanishing spinor defines a parallelism, so the well-known fact that a 3-manifold is parallelisable if and only if it is orientable (equivalently spin) can be viewed as analogous to the existence of G_2-structures spin 7-manifolds.

Remark 4.3. Note the distinction between parallelisms (trivialisations of the tangent bundle itself) and framings, *i.e.* trivialisations of the direct sum of TM with a trivial bundle. Given a parallelism on a 3-manifold M, other framings correspond to maps $M \to Spin(3)$, and framings to maps $M \to Spin$ to the stable spin group. Up to homotopy, these are classified by $H_3(Spin(3))$ and $H_3(Spin)$ respectively. Both are isomorphic to $\mathbb{Z}$, but the natural map $H_3(Spin(3)) \to H_3(Spin)$ has cokernel $\mathbb{Z}_2$. For example, the standard framing of B^4 restricts to a framing of S^3 that is not homotopic to a parallelism.

Let t be a parallelism of a 3-manifold M and W a spin coboundary. As above, let $n_+(W,t)$ denote the intersection number with the zero section of a positive spinor field on W restricting to the spinor defining t on M. Then (14) implies that $-4n_+(W,t) + 3\sigma(W) + 2\chi(W) \in \mathbb{Z}$ is independent of the choice of W. This invariant distinguishes between the homotopy classes of parallelisms on M. (Because this is a complete $\mathbb{Z}$-valued invariant, the diffeomorphisms of M must act trivially on the set of homotopy classes of parallelisms.)

Since $\sigma(X) = -8\widehat{A}(X)$ for a closed spin 4-manifold, (14) is equivalent to

$$12\widehat{A} = -2e_\pm \pm e \tag{16}$$

for any $Spin(4)$-bundle. Therefore an alternative way to define an invariant of a parallelism t on a 3-manifold that is an analogue to ν is to consider $\nu'(t) = -2n_+(W,t) + \chi(W) \in \mathbb{Z}_{24}$. This is in fact equivalent to Adams' e-invariant [1, §7]. For t can be used to define a vector bundle TW/t on the quotient topological space W/M. Now $-2n_+(W,t) + \chi(W)$ is $-2e_+(TW/t) + e(TW/t)$ evaluated on the generator of $H_4(W/M)$. By (16) this equals $12\widehat{A}(TW/t) \mod 24$, and the Adams invariant of t can be defined as $\frac{1}{2}\widehat{A}(TW/t) \mod \mathbb{Z}$ (see *e.g.* [2, (4.11)]).

Note that because $\widehat{A}$ is a stable class, the Adams invariant can be defined in the same way for (stable) framings of M, and it is clear that it is an invariant under (stably) framed bordism; indeed, it is well-known that it realises an isomorphism $\Omega_3^{\mathrm{fr}} \cong \mathbb{Z}_{24}$. The generator is S^3 with the Lie group framing t_{rd}. Note that this is the parallelism induced by the flat $SU(2)$-structure on B^4, hence $\nu'(t_{rd}) = 1$.

On the other hand, as $\nu'(t)$ was defined in terms of unstable classes, it is only defined when t is parallelism. $\nu'(t) = \chi(W) \mod 2$ clearly depends only on the spin structure defined by t and not on the parallelism itself, and (15) interprets it intrinsically. In particular, if $\chi_{\mathbb{Q}}(M) + GS(M)$ is odd, then no parallelism on M has a parallelised coboundary. (This argument does not apply to (stable) framings on M).

We can use the interpretation of parallelisms in terms of spinors to show that the parallelised bordism group is also $\mathbb{Z}_{24}$, generated by (S^3, t_{rd}). For any parallelism t on M we argued in the proof of Lemma 4.2 that there exists an $SU(2)$-coboundary W. Choose a vector field V on W that is outward-pointing at M and has transverse zeros. Let W' be the result of cutting out a ball from W near each zero of V. Then V together with the $SU(2)$-structure defines a parallelism of W', which restricts to t_{rd} or its inverse on each S^3 component of $\partial W' \setminus M$. This proves that t_{rd} generates the bordism group. We already observed that $\nu'(t_{rd}) = 1$, so to see that t_{rd} has order 24 it suffices to note that $K3$ has an $SU(2)$-structure and a vector field with 24 zeros.

4.4. **Twisted connected sums.** Now we sketch the basics of the twisted connected sum construction, ignoring many details that are required to justify that the resulting G_2-structures are torsion-free. The construction starts from a pair of asymptotically cylindrical Calabi-Yau 3-folds $V_\pm$. We can think of these as a pair of simply connected 6-manifolds with boundary $S^1 \times D_\pm$, for $D_\pm$ a K3 surface. They are equipped with $SU(3)$-structures $(\omega_\pm, \Omega_\pm)$ such that on a collar neighbourhood $C_\pm \cong [0,1) \times \partial V_\pm$ of the boundary

$$\begin{aligned} \omega_\pm &= dt \wedge d\vartheta + \omega_\pm^I, \\ \Omega_\pm &= (d\vartheta - i dt) \wedge (\omega_\pm^J + i\omega_\pm^K), \end{aligned}$$

where ϑ is the S^1-coordinate and t is the collar coordinate. The construction assumes that there is a diffeomorphism $f : D_+ \to D_-$ such that $f^*\omega_-^I = \omega_+^J$, $f^*\omega_-^J = \omega_+^I$ and $f^*\omega_-^K = -\omega_+^K$. Now define G_2-structures on $S^1 \times V_\pm$ by

$$\varphi_\pm = d\theta \wedge \omega_\pm + \operatorname{Re}\Omega_\pm,$$

where θ denotes the S^1-coordinate, and a diffeomorphism

$$\begin{aligned} F:\ \partial(S^1 \times V_+) \cong S^1 \times S^1 \times D_+ \ &\longrightarrow\ S^1 \times S^1 \times D_- \cong \partial(S^1 \times V_-), \\ (\theta, \vartheta, x) \ &\longmapsto (\vartheta, \theta, f(x)). \end{aligned}$$

In the collar neighbourhoods $C_\pm$

$$\varphi_\pm = d\theta \wedge dt \wedge d\vartheta + d\theta \wedge \omega_\pm^I + d\vartheta \wedge \omega_\pm^J + dt \wedge \omega_\pm^K,$$

so φ_+ and φ_- patch up to a well-defined G_2-structure φ on the closed manifold

$$M = (S^1 \times V_+) \cup_F (S^1 \times V_-). \tag{17}$$

Up to perturbation, this G_2-structure is torsion-free. Because F swaps the circle factors at the boundary, M is simply-connected.

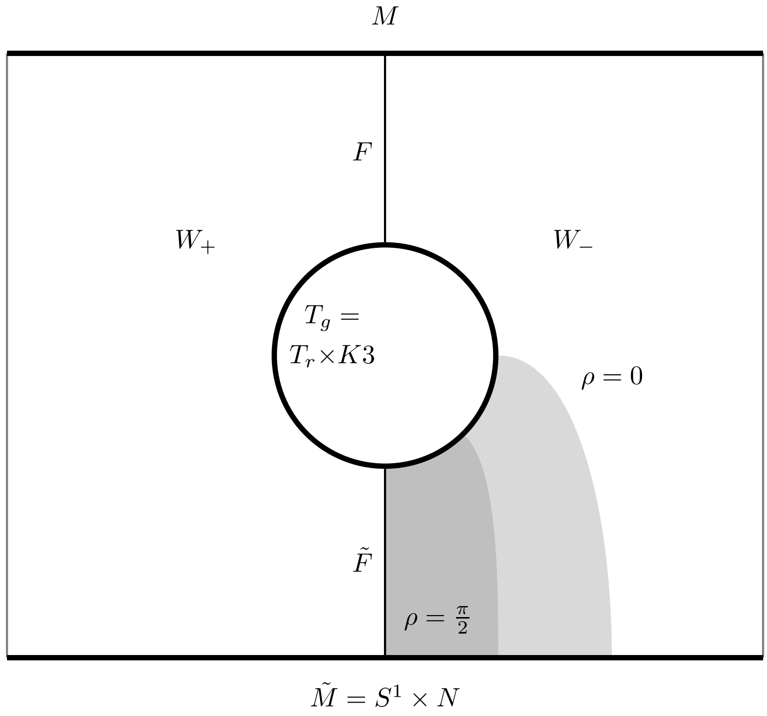

FIGURE 1. The 'keyhole' bordism W

4.5. **A** $Spin(7)$**-bordism.** We now proceed with the proof of Theorem 1.10, that any twisted connected sum G_2-manifold has $\nu = 24$. Consider the diffeomorphism

$$\tilde{F} = id \times -id \times f : S^1 \times S^1 \times D_+ \to S^1 \times S^1 \times D_-,$$

and the "untwisted connected sum" $\tilde{M} = (S^1 \times V_+) \cup_{\tilde{F}} (S^1 \times V_-)$. Then $\tilde{M} = S^1 \times N$, where $N = V_+ \cup_{-id\times f} V_-$. If we let r denote the right angle rotation $(\theta, \vartheta) \mapsto (\vartheta, -\theta)$ of $S^1 \times S^1$, then $g = F \circ \tilde{F}^{-1} = r \times id_{K3}$. Its mapping torus T_g is therefore $T_r \times K3$, where T_r denotes the mapping torus of r.

To compute $\nu(\varphi)$ of the twisted connected sum G_2-structure φ on M and prove Theorem 1.10 we will construct a $Spin(7)$-bordism W to product G_2-structures on $\tilde{M} \sqcup T_g$. Let

$$B_\pm = \left\{(y - \tfrac{1}{2})^2 + t^2 < \tfrac{1}{4}\right\} \subset I \times S^1 \times C_\pm,$$
$$W_\pm = I \times S^1 \times V_\pm \setminus B_\pm,$$

where y denotes the I-coordinate, and t the collar coordinate on $C_\pm \subset V_\pm$ as before. $\partial W_\pm$ is a union of five pieces, meeting in edges at $\{y\} \times S^1 \times S^1 \times K3$ for $y = 0, \frac{1}{4}, \frac{3}{4}$ and 1: a 'top' and 'bottom' piece each diffeomorphic to $S^1 \times V_\pm$, $[0, \frac{1}{4}] \times S^1 \times S^1 \times D_\pm$ and $[\frac{3}{4}, 1] \times S^1 \times S^1 \times D_\pm$, and $E_\pm := \left\{(y - \frac{1}{2})^2 + t^2 = \frac{1}{4}\right\} \subset I \times S^1 \times C_\pm$.

We form a 'keyhole' bordism W by gluing some of these pieces: identify $[0, \frac{1}{4}] \times S^1 \times S^1 \times D_\pm$ via $id \times \tilde{F}$, and $[\frac{3}{4}, 1] \times S^1 \times S^1 \times D_\pm$ via $id \times F$. Then ∂W is a disjoint union $M \sqcup \tilde{M} \sqcup T_g$, where M is formed by gluing the top pieces of ∂W_+ and ∂W_- and $\tilde{M}$ by gluing the bottom pieces, while the keyhole boundary component $E_+ \cup E_-$ can be identified with the mapping torus T_g.

It is easy to compute that $H_1(T_r) \cong \mathbb{Z} \times \mathbb{Z}_2$. Therefore $\chi_{\mathbb{Q}}(T_r) \equiv 0$ and $GS(T_r) = \pm 1$. Since $\widehat{A}(K3) = 2$, Lemma 4.2 implies that any product G_2-structure on $T_r \times K3$ has $\nu = 24$, while a product G_2-structure on $\tilde{M}$ has $\nu = 0$. To complete the calculation of $\nu(\varphi)$ it remains to compute the topological invariants of the $Spin(7)$-bordism W.

Lemma 4.4. $\sigma(W) = \chi(W) = 0$.

Proof. For the Euler characteristic, we use the usual inclusion-exclusion formula. The manifolds W_+, W_- and $W_+ \cap W_-$ all contain S^1 factors, so $\chi(W) = \chi(W_+) + \chi(W_-) - \chi(W_+ \cap W_-) = 0$.

For the signature, we must apply Wall's signature formula [23] because W is formed by gluing W_+ and W_- along only parts of boundary components. The piece of the boundaries of W_+ and W_- that we glue is $X_0 = ([0, \frac{1}{4}] \sqcup [\frac{3}{4}, 1]) \times T^2 \times K3$. Let $Z = \partial X_0 = \left\{0, \frac{1}{4}, \frac{3}{4}, 1\right\} \times T^2 \times K3$ (the edges of $\partial W_\pm$), and

$$X_\pm = \partial(W_\pm) \setminus X_0 = \{0,1\} \times S^1 \times V_\pm \ \sqcup \ E_\pm.$$

We need to identify the images A, B and C in $H^3(Z)$ of $H^3(X_0)$, $H^3(X_+)$ and $H^3(X_-)$, respectively (each is a Lagrangian subspace). The vectors space $K = \frac{A\cap(B+C)}{(A\cap B)+(A\cap C)}$ admits a natural symmetric bilinear form, and since $W_\pm$ both have signature 0, the signature formula states that the signature of W equals the signature of K.

The group $H^3(Z)$ consists of 8 copies of the $K3$ lattice L, and we can label them $L_{y\pm}$ depending on the value of $y \in \left\{0, \frac{1}{4}, \frac{3}{4}, 1\right\}$ on the corresponding component of Z, and whether it comes from the ϑ or θ factor in T^2. $L_\pm = \bigoplus_y L_{y\pm}$ are Lagrangian subspaces of $H^3(Z)$. Let $N_\pm$ denote the image of $H^2(V_\pm)$ in $H^2(D_\pm) \cong L$, and $T_\pm \subset L$ the orthogonal complement. Then

$$\begin{aligned}
A &= \{c_{0\pm} = c_{\frac{1}{4}\pm},\ c_{\frac{3}{4}\pm} = c_{1\pm}\} \\
B &= \{c_{0+}, c_{1+} \in N_+,\ c_{0-}, c_{1-} \in T_+,\ c_{\frac{1}{4}\pm} = c_{\frac{3}{4}\pm}\} \\
C &= \{c_{0-}, c_{1+} \in N_+,\ c_{0+}, c_{1-} \in T_-,\ c_{\frac{1}{4}\pm} = c_{\frac{3}{4}\pm}\}
\end{aligned}$$

Notice that each of A, B and C is a direct sum of its intersections with L_+ and L_-. It follows that K splits as a direct sum of elements represented by L_+ and L_-, and each of the summands is isotropic. Therefore $\sigma(W) = \sigma(K) = 0$. □

To finish the proof of Theorem 1.10, we need to exhibit a *Spin*(7)-structure on W with the right restrictions to the boundary components. There is no natural Calabi-Yau structure on N. We can, however, define an $SU(3)$-structure as follows. Let V'_- be the complement of the collar neighbourhood $C_- \subset V_-$. On C_- set

$$\begin{aligned}
\omega' &= dt \wedge d\vartheta + c_\rho \omega_-^I + s_\rho \omega_-^J, \\
\Omega' &= (d\vartheta - i dt) \wedge (c_\rho \omega_-^J - s_\rho \omega_-^I + i\omega_-^K),
\end{aligned}$$

where $c_\rho = \cos\rho$, $s_\rho = \sin\rho$ for a smooth function ρ supported on C_-, such that $\rho = \frac{\pi}{2}$ on ∂V_-. Take $\tilde{\omega}$ to be ω_+ on V_+, ω' on C_-, and ω_- on V'_-, and define $\tilde{\Omega}$ analogously. Then $(\tilde{\omega}, \tilde{\Omega})$ is a well-defined $SU(3)$-structure on N, and $\tilde{\varphi} = d\theta \wedge \tilde{\omega} + \operatorname{Re}\tilde{\Omega}$ is a product G_2-structure on $\tilde{M}$.

Next we define the *Spin*(7)-structure ψ on W. Let y be the I coordinate on each half. First, define ρ on $I \times C_-$ to be $\frac{\pi}{2}$ on a neighbourhood of $[0, \frac{1}{4}] \times \partial V_-$ and have compact support in $\left[0, \frac{1}{2}\right) \times C_-$ (see Figure 1), and use this to define forms $\tilde{\omega}$ and $\tilde{\Omega}$ on $I \times V_-$. Since dy is a global covector field on W_0, defining a *Spin*(7)-structure is equivalent to defining a G_2-structure on each slice $y = \text{const}$. Take this to be $\varphi_+ = d\theta \wedge \omega_+ + \operatorname{Re}\Omega_+$ on $\{y\} \times S^1 \times V_+$, and $d\theta \wedge \tilde{\omega} + \operatorname{Re}\tilde{\Omega}$ on $\{y\} \times S^1 \times V_-$. Then the restriction of ψ to the boundary components M and $\tilde{M}$ are φ and $-\tilde{\varphi}$ respectively, as desired.

Now we show that the restriction of ψ to the 'keyhole' boundary component $T_g = E_+ \cup E_-$ is a product G_2-structure too. An abbreviated justification starts from $E_\pm \cong I \times S^1 \times S^1 \times D_\pm$ being embedded as a product inside $I \times C_\pm$. The restriction of ψ to $I \times C_\pm$ is a product of two $SU(2)$-structures, so the induced G_2-structure on $E_\pm$ is a product of a coframe field on $I \times S^1 \times S^1$ and an $SU(2)$-structure on $K3$. The coframes on the two copies of $I \times S^1 \times S^1$ then patch up to a coframe on their union T_r, and the G_2-structure on T_g is the product of that with an $SU(2)$-structure on $K3$.

Writing down the structures explicitly is rather cumbersome. To make the notation slightly more manageable we will use a complex form as a shorthand for an ordered pair of real forms, so that an $SU(2)$-structure can be defined by one complex and one real 2-form, or a coframe field on a 3-manifold by one complex and one real 1-form. Also, we identify both D_+ and D_- with

a standard $K3$, so that f corresponds to id_{K3}. Setting $y = -\frac{1}{2}c_\alpha + \frac{1}{2},\ t = \frac{1}{2}s_\alpha$ for $\alpha \in [0,\pi]$ lets us identify $E_+ \subset I \times C_+$ with $[0,\pi] \times S^1 \times S^1 \times K3$. On $I \times C_+$, ψ is the product of the $SU(2)$-structure

$$\big((dy - idt) \wedge (d\theta + id\vartheta),\ dy \wedge dt - d\theta \wedge d\vartheta\big) \tag{18}$$

on $I \times [0,1) \times S^1 \times S^1$ and $(\omega_+^I + i\omega_+^J,\ \omega_+^K)$ on $K3$. The induced G_2-structure on E_+ is given by contraction with the normal vector field $c_\alpha \frac{\partial}{\partial y} - s_\alpha \frac{\partial}{\partial t}$. The result is the product of the same $SU(2)$-structure on $K3$ with the coframe field $(e^{i\alpha}(d\theta + id\vartheta),\ \frac{1}{2}d\alpha)$ on $[0,\pi] \times S^1 \times S^1$.

Similarly, for $\alpha \in [\pi, 2\pi]$ we set $y = -\frac{1}{2}c_\alpha + \frac{1}{2},\ t = -\frac{1}{2}s_\alpha$ to identify $[\pi,2\pi] \times S^1 \times S^1 \times D_- \cong E_-$. On $I \times C_-$, the restriction of ψ is given by the product of (18) on $I \times [0,1) \times S^1 \times S^1$ and $\big(e^{-i\rho}(\omega_-^I + i\omega_-^J),\ \omega_-^K\big)$ on the tangent space to the $K3$ factor. Contracting with the normal vector field $c_\alpha \frac{\partial}{\partial y} + s_\alpha \frac{\partial}{\partial t}$ gives the coframe $\big(e^{-i\alpha}(d\theta + id\vartheta),\ -\frac{1}{2}d\alpha\big)$ on $[\pi, 2\pi] \times S^1 \times S^1$. Now, as product G_2-structures

$$\begin{aligned}&\big(e^{-i\alpha}(d\theta + id\vartheta),\ -\tfrac{1}{2}d\alpha\big) \cdot \big(e^{-i\rho}(\omega_-^I + i\omega_-^J),\ \omega_-^K\big) = \\ &\Big(e^{i(\rho-\alpha)}(d\theta + id\vartheta),\ -\tfrac{1}{2}d\alpha\Big) \cdot (\omega_-^I + i\omega_-^J,\ \omega_-^K) = \Big(e^{i(\alpha-\rho)}(d\vartheta + id\theta),\ \tfrac{1}{2}d\alpha\Big) \cdot (\omega_+^I + i\omega_+^J,\ \omega_+^K).\end{aligned}$$

T_g is formed by gluing boundaries of $[0,\pi] \times S^1 \times S^1 \times K3$ and $[\pi, 2\pi] \times S^1 \times S^1 \times K3$ using $(\pi, \theta, \vartheta, x) \mapsto (\pi, \vartheta, \theta, x)$ and $(0, \theta, \vartheta, x) \mapsto (2\pi, \vartheta, -\theta, x)$. These maps preserve the $SU(2)$-structure on the $K3$ factor, and match up the coframes $(e^{i\alpha}(d\theta + id\vartheta),\ \frac{1}{2}d\alpha)$ and $\big(e^{i(\alpha-\rho)}(d\vartheta + id\theta),\ \frac{1}{2}d\alpha\big)$ to a well-defined coframe on T_r (since $\rho = 0$ at $\alpha = \pi$ and $\rho = \frac{\pi}{2}$ at $\alpha = 0,\ 2\pi$). Thus the G_2-structure on $T_g = T_r \times K3$ is a product, completing the proof of Theorem 1.10

4.6. **Orbifold resolutions.** For some of Joyce's examples of compact G_2-manifolds constructed by resolving flat orbifolds, the torsion-free G_2-structures are homotopic to twisted connected sum G_2-structures, and thus have $\nu = 24$. It is proved in [16] that in some cases there is even a connecting path of torsion-free G_2-structures, but that is of course of no importance for the calculation of ν.

We have no general technique for computing ν of orbifold resolution G_2-manifolds. We note, however, that a small number of examples have $b_2(M) + b_3(M)$ even, *e.g.* [13, §12.8.4]. Those G_2-manifolds have $\chi_{\mathbb{Q}}(M)$ and hence ν odd.

5. The action of spin diffeomorphisms on $\mathcal{G}_2^h(M)$

Recall that (M, φ) is a closed connected spin 7-manifold with G_2-structure. In this section we investigate the action of the group of spin diffeomorphisms of M on the set of homotopy classes of G_2-structures on M:

$$\mathcal{G}_2^h(M) \times \mathrm{Diff}_{Spin}(M) \to \mathcal{G}_2^h(M), \quad ([\varphi], f) \mapsto [f^*\varphi].$$

To determine this action for a specific spin diffeomorphism $f\colon M \cong M$ amounts to computing difference class $D(\varphi, f^*\varphi)$. The existence of the ν-invariant ensures that $D(\varphi, f^*\varphi) = 24k$ for some integer k. In this section we relate the possible values of k to the topology of M and in particular $p_M \in H^4(M)$. We begin with some necessary preliminaries about the elementary algebra of elements in abelian groups before moving to the topology.

5.1. **Divisibilities of elements of abelian groups.** In this subsection we define the positive integers $d_\pi(M)$ and $d_\infty(M)$ used in the statement of Theorem 1.13. Let G be a finitely generated abelian group with identity element 0, for example $G = H^4(M)$. We call an element $x \in G$ *primitive* if $x \neq 0$ and if the cyclic group generated by x is a summand of G. For a general element $x \in G$ we define the divisibility of x, $d(x)$, as follows:

$$d(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \text{ is primitive} \\ \mathrm{Max}\{r \in \mathbb{Z} \,|\, x = ry,\ y \in G \text{ and } y \text{ is primitive}\} & \text{otherwise} \end{cases}$$

Let $T \subset G$ be the torsion subgroup and let $\pi\colon G \to F := G/T$ be the projection to the free quotient of G. We define the positive integer

$$d_\pi(x) := d(\pi(x))$$

and for a spin 7-manifold M the positive integer (even by Lemma 2.6)

$$d_\pi(M) := d_\pi(p_M) \in 2 \cdot \mathbb{Z}.$$

Following the formulation of [25, Conjecture p. 48], for $x \in G$ we next define the positive integer

$$d_\infty(x) := \mathrm{Max}\{r \mid r, N \in \mathbb{Z},\ rN^2 \text{ divides } Nx\}.$$

We remark that we have the following chain of divisibilities

$$d(x) \mid d_\infty(x) \mid d_\pi(x)$$

and that $d(x) = d_\pi(x)$ if and only if $d_\infty(x) = d_\pi(x)$. For a spin 7-manifold M we define the positive even integer

$$d_\infty(M) := d_\infty(p_M) \in 2 \cdot \mathbb{Z}.$$

Example 5.1. Let $\alpha \in \pi_3(SO_3) \cong \mathbb{Z}$, let $S\alpha \in \pi_3(SO_4)$ be its stabilisation and let $M = S^3 \tilde{\times}_{S\alpha} S^4$ be the total space of the sphere bundle associated to $S\alpha$. Then by [17] $(H^4(M), p_M) \cong (\mathbb{Z}, \alpha)$ and so $d(p_M) = d_\pi(M) = d_\infty(M) = 2\alpha$.

On the other hand we have the spin 7-manifolds $M = P_1 \sharp P_2$ from [25, §6] for which we have $(H^4(M), p_M) \cong (\mathbb{Z} \oplus \mathbb{Z}_7, 7 \cdot 2^k \oplus 2)$ for $k \geq 4$. In this case $d_\pi(M) = 7 \cdot 2^k$ but $d(p_M) = d_\infty(M) = 2^k$.

We conclude this subsection by recording an alternative expression for $d_\infty(x)$ which is sheds some light on its definition. There is a split, but not canonically split, short exact sequence

$$0 \to T \to G \xrightarrow{\pi} F \to 0.$$

Observe for all retractions $s\colon G \to T$ over the inclusion $T \to G$ that the element $[s(x)] \in T/d_\pi(x)T$ is well-defined. Define

$$e(x) := \text{the order of } [s(x)] \in T/d_\pi(x)T.$$

The proof the following lemma is left as an exercise for the reader.

Lemma 5.2. *For all $x \in G$ we have $d_\infty(x) = d_\pi(x)/e(x)$.*

5.2. **Translations of G_2-structures and mapping tori.** Given (M, φ) and a spin diffeomorphism $f\colon M \cong M$, we wish to calculate the difference element $D(\varphi, f^*\varphi) \in \mathbb{Z}$. We first establish that $D(\varphi, f^*\varphi)$ depends only on the pseudo-isotopy class of f. Recall that a pseudo-isotopy between spin diffeomorphisms f_0 and f_1 is a diffeomorphism $F\colon M \times I \cong M \times I$ where $F|_{M \times \{i\}} = f_i$ for $i = 0, 1$. Now extend the defining spinor of φ to a translation-invariant positive spinor field on $M \times I$. Pulling back this extended spinor by a pseudo-isotopy $F\colon M \times I \cong M \times I$ gives a non-zero spinor that interpolates between $f_1^*\varphi$ and $-f_0^*\varphi$, where $f_i := F|_{M \times \{i\}}$. Hence $f_0^*\varphi$ and $f_1^*\varphi$ are homotopic and so we obtain an integer valued function

$$D_M\colon \widetilde{\pi}_0 \mathrm{Diff}_{Spin}(M) \to \mathbb{Z}, \quad [f] \mapsto D(\varphi, f^*\varphi),$$

where $\widetilde{\pi}_0 \mathrm{Diff}_{Spin}(M)$ denotes the group of pseudo-isotopy classes of spin diffeomorphisms of M. We point out that Lemma 5.3 below justifies the notation since D_M does not depend upon the G_2-structure φ.

The integer $D_M(f)$ measures the translation action of f on the set of homotopy classes of G_2-structures. Next we show how to calculate $D_M(f)$ using the mapping torus of f:

$$T_f := (M \times [0,1])/(x, 0) \sim (f(x), 1).$$

Since f is a spin diffeomorphism the closed 8-manifold T_f admits a spin structure. We choose a spin structure and let T_f to denote the corresponding 8-dimensional spin manifold: no confusion shall arise since we are interested only in the characteristic number

$$p^2(f) := \langle p_{T_f}^2, [T_f] \rangle \in \mathbb{Z}$$

which depends only on the oriented diffeomorphism type of T_f since $2p_{T_f} = p_1(T_f)$ and $H^8(T_f) \cong \mathbb{Z}$ (in fact p_{T_f} is independent of the choice of spin structure by [5, p. 170]). We see that the integer $p^2(f)$ depends only on the pseudo-isotopy class of f and we may therefore define the function

$$p^2 \colon \widetilde{\pi}_0 \mathrm{Diff}_{Spin}(M) \to \mathbb{Z}, \quad [f] \mapsto p^2(f).$$

The following proposition proves Proposition 1.12 and shows how the mapping torus T_f can be used to compute the difference class $D(\varphi, f^*\varphi)$.

Proposition 5.3. *The function $D_M \colon \widetilde{\pi}_0 \mathrm{Diff}_{Spin}(M) \to \mathbb{Z}$ is a homomorphism given by the equation*

$$D(\varphi, f^*\varphi) = \frac{-3 \cdot p^2(f)}{28} = -24\widehat{A}(T_f).$$

Proof. From the definition of $D(\varphi, \varphi')$ in §3 it is clear that $D(f^*\varphi, f^*\varphi') = D(\varphi, \varphi')$ for any spin diffeomorphism f and any pair of G_2-structures φ and φ' on M. Now for two spin diffeomorphisms $f_0, f_1 \colon M \cong M$, by Lemma 1.7 we have

$$D(\varphi, (f_1 \circ f_0)^*\varphi) = D(\varphi, f_0^*\varphi) + D(f_0^*\varphi, f_0^*(f_1^*\varphi)) = D(\varphi, f_0^*\varphi) + D(\varphi, f_1^*\varphi).$$

This shows that D_M is a homomorphism.

Turning to the mapping torus, from (11) and the discussion preceding it, we see that the difference class $D(\varphi, f^*\varphi)$ may be computed by taking the $Spin(7)$-bordism

$$W_f := (M \times [0,1]) \cup_f (M \times [1,2])$$

between M and $-M$ where we glue two copies of $M \times I$ together using f. Clearly W_f is a $Spin(7)$-bordism between φ and $f^*\varphi$. We may identify the mapping torus T_f with the manifold

$$\overline{W}_f = W_f \cup_{\mathrm{Id}_M \sqcup \mathrm{Id}_M} (M \times I) \tag{19}$$

and (6) gives

$$D(\varphi, f^*\varphi) = -e_+(\overline{W}_f) = -e_+(T_f).$$

By Proposition 2.3, $e_+(T_f) = \frac{1}{16}(4p_{T_f}^2 - 4p_2 + 8e)$ and using the signature theorem to eliminate p_2 from this equation we have

$$D(\varphi, f^*\varphi) = -e_+(T_f) = \frac{-3p_{T_f}^2}{28} + \frac{45\sigma(T_f)}{28} - \frac{\chi(T_f)}{2}.$$

Since T_f is a mapping torus both $\sigma(T_f)$ and $\chi(T_f)$ vanish which proves the first equality of the proposition. The second equality now follows from the following equation for the $\widehat{A}$ genus of a closed spin 8-manifold X:

$$\widehat{A}(X) = \frac{1}{2^5 \cdot 7}\left(p_X^2 - \sigma(X)\right). \tag{20}$$

Equation (20) is easily deduced from the expressions for the L-genus and the $\widehat{A}$-genus given in the proof of Corollary 2.4. It was already established for example in [10, §6] □

5.3. **Constraints on translations of G_2-structures.** In this subsection we prove Theorem 1.13 (i) and (ii) which establish lower bounds on the possible translations values of $D(\varphi, f^*\varphi)$ for any spin diffeomorphism $f \colon M \cong M$. The following lemma restates Theorem 1.13 (i) and (ii).

Lemma 5.4. *Let M be a closed spin 7-manifold. Then*

$$D_M\big(\widetilde{\pi}_0 \mathrm{Diff}_{Spin}(M)\big) \subseteq 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{224}\right) \cdot \mathbb{Z}. \tag{21}$$

If $H^4(M)$ has no 2-torsion then

$$D_M\big(\widetilde{\pi}_0 \mathrm{Diff}_{Spin}(M)\big) \subseteq 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{112}\right) \cdot \mathbb{Z}. \tag{22}$$

We shall use the following simple lemma to prove Lemma 5.4.

Lemma 5.5. *Let T_f be the mapping torus of $f \colon M \cong M$ and $i \colon M \to T_f$ the inclusion.*

(i) *If $x \in H^4(T_f)$ is such that s divides i^*x then s divides $x^2 \in H^8(T_f) \cong \mathbb{Z}$.*

(ii) *If in addition the torsion in $H^4(M)$ is odd and s is even then $2s$ divides x^2.*

Proof. (i) Consider the following fragment of the long exact cohomology sequence for the mapping torus T_f with $\mathbb{Z}_s$ coefficients:

$$\to H^3(M;\mathbb{Z}_s) \xrightarrow{\mathrm{Id}-f^*} H^3(M;\mathbb{Z}_s) \xrightarrow{\partial} H^4(T_f;\mathbb{Z}_s) \xrightarrow{i^*} H^4(M;\mathbb{Z}_s) \xrightarrow{\mathrm{Id}-f^*} H^4(M;\mathbb{Z}_s) \to$$

For a space X, let $\rho_s\colon H^*(X) \to H^*(X;\mathbb{Z}_s)$ denote reduction mod s. By assumption $i^*\rho_s(x) = 0$ and so $\rho_s(x)$ lies in the image of ∂. But the cup-product

$$H^4(T_f;\mathbb{Z}_s) \times H^4(T_f;\mathbb{Z}_s) \to \mathbb{Z}_s$$

vanishes on $\mathrm{Im}(\partial)$. Hence $\rho_s(x)^2 = \rho_s(x^2) = 0 \in H^8(T_f;\mathbb{Z}_s)$ and so s divides x^2.

(ii) We first factorise $s = 2^k s'$ where s' is odd, and $k \geq 1$ by hypothesis. By part (i) we know that s' divides x^2 so we must show that 2^{k+1} divides x^2 as well. If the torsion in $H^4(M)$ is odd then $H^3(M) \to H^3(M;\mathbb{Z}_{2^k})$ is surjective. The argument above therefore implies that there is a $z \in H^3(M)$ such that $x - \partial(z)$ is divisible by 2^k, say equal to $2^k y$. Then

$$x^2 = (2^k y + \partial(z))^2 = 2^k(2^k y^2 + 2y\partial(z)),$$

which is divisible by 2^{k+1}. □

Proof of Lemma 5.4. From the definition of $d_\infty(M) = d_\infty(p_M)$ there is a positive integer N such that $d_\infty(M)N^2$ divides Np_M. Applying Lemma 5.5(i) with $x = Np_{T_f}$ and $s = d_\infty(M)N^2$ gives that $d_\infty(M)N^2$ divides $N^2 p_{T_f}^2$ and hence

$$p_{T_f}^2 \in d_\infty(M) \cdot \mathbb{Z}. \tag{23}$$

Now we turn to the factor of $8 \cdot 28$ in $p^2(f)$. From (20) we see that for 8-dimensional spin manifold X we have

$$\sigma(X) - p_X^2 \in 8 \cdot 28 \cdot \mathbb{Z}.$$

Since the spin mapping torus T_f is a closed 8-dimensional spin manifold with $\sigma(T_f) = 0$ we deduce that

$$p_{T_f}^2 \in 8 \cdot 28 \cdot \mathbb{Z}. \tag{24}$$

Combining (23) and (24) we conclude that $p_{T_f}^2 \in \mathrm{lcm}(d_\infty(M), 224) \cdot \mathbb{Z}$. Applying Lemma 5.3 gives the containment (21).

Similarly, if $H^4(M)$ has no 2-torsion, then it follows from Lemma 5.5(ii) that $p_{T_f}^2 \in 2d_\infty(M) \cdot \mathbb{Z}$ (since we know p_M is even). Combining with (24) gives $p_{T_f}^2 \in \mathrm{lcm}(2d_\infty(M), 224) \cdot \mathbb{Z}$. Applying Lemma 5.3 gives the containment (22). □

5.4. **Realising translations of G_2-structures.** In this subsection we construct diffeomorphisms of certain spin 7-manifolds and thereby prove Theorem 1.13 (iii) which follows very quickly from the following lemma.

Lemma 5.6. *Suppose that M is a closed spin 7-manifold such that there is a diffeomorphism $M \cong M_0 \sharp N$ where M_0 is 2-connected. Then*

$$D_M\big(\widetilde{\pi}_0 \mathrm{Diff}_{Spin}(M)\big) \supseteq 24 \cdot \mathrm{Num}\left(\frac{d_\pi(M_0)}{112}\right) \cdot \mathbb{Z}.$$

Proof of Theorem 1.13 (iii). We suppose that $M = N \sharp M_0$ and that φ and φ' are G_2 structures on M such that $D(\varphi, \varphi') = 24 \cdot \mathrm{Num}\left(\frac{d_\pi(M_0)}{112}\right) j$ for some integer j. By Lemma 5.6 there is a spin diffeomorphism $g\colon M \cong M$ such that $D(\varphi, g^*\varphi) = D(\varphi, \varphi')$. Hence by Lemma 1.7, $D(\varphi', g^*\varphi) = 0$ and so φ' and $g^*\varphi$ are homotopic. □

Proof of Lemma 5.6. We identify $M = M_0 \sharp N$ where M_0 is 2-connected and for convenience we abbreviate $d_\pi(M_0) = d$. By [24, Theorem 1] we may decompose M_0 as a connected sum

$$M_0 \cong_{Spin} M_1 \sharp M_2$$

where $d_\pi(M_1) = d$ and M_1 is the total space of a certain 3-sphere bundle over S^4 with Euler class zero as in Example 5.1. Specifically, there is a linear D^3-bundle with characteristic map

$\alpha \in \pi_3(SO_3)$ such that $M_1 = S^3 \tilde{\times}_{S\alpha} S^4$ is the total space of the sphere bundle of the stabilisation of α, $S\alpha \in \pi_3(SO_4)$ and $p_{M_1} = p(\alpha) = d \cdot z$ where z is a generator of $H^4(M_1; \mathbb{Z})$. We shall produce the required diffeomorphisms on the manifold M_1 and then extend by the identity to M_0 and then M. Let

$$M_1^\bullet := M_1 - \mathrm{Int}(D^7)$$

be M_1 minus a small open disc. Since M_1 is the total space of an S^3-bundle over S^4 there is a diffeomorphism

$$M_1^\bullet \cong (D^3 \tilde{\times}_\alpha S^4) \cup_{S^2 \times D^4} (D^3 \times D^4)$$

where $D^3 \tilde{\times} S^4$ is a tubular neighbourhood of a section of $M_1 \to S^4$ and $D^3 \times D^4$ is a 3-handle.

By [22, p. 171 (2)] we may identify $\pi_3(SO_4)$ as the group of pairs of integers (n, p) where $n \equiv p$ mod 2 so that the corresponding bundle over S^4 has Euler class $n \in H^4(S^4; \mathbb{Z}) = \mathbb{Z}$ and first Pontrjagin class $2p$. Let $\gamma_{n,p} \colon (D^3, S^2) \to (SO_4, \mathrm{Id})$ be a smooth function representing (n, p). We define a diffeomorphism

$$f^\bullet_{n,p} \colon M_1 \cong M_1$$

where $f^\bullet_{n,p}|_{D^3 \tilde{\times} S^4}$ is the identity and on the 3-handle we use the D^3 co-ordinate to twist the D^4-co-ordinate using $\gamma_{n,p}$. To be explicit:

$$f_{n,p}|_{D^3 \times D^4}(u, v) = (u, \gamma_{n,p}(u)(v)).$$

To see if we can extend $f^\bullet_{n,p}$ to M_1 we need to compute the pseudo-isotopy class of the induced diffeomorphism $\partial f^\bullet_{n,p} \colon S^6 \cong S^6$. By [21, 7, 14], there are isomorphisms

$$\widetilde{\pi}_0 \mathrm{Diff}_+(S^6) \cong \Theta_7 \cong \mathbb{Z}_{28}$$

where $\widetilde{\pi}_0 \mathrm{Diff}_+(S^6)$ is the group of pseudo-isotopy classes of orientation preserving diffeomorphisms of the 6-sphere. We compute $[\partial f^\bullet_{n,p}] \in \mathbb{Z}/28$ as follows. The manifold $M_1 \cong S^3 \tilde{\times}_{S\alpha} S^4$ bounds the 8-dimensional D^4-bundle $W_0 := D^4 \tilde{\times}_{S\alpha} S^4$. Form the manifold 8-manifold with boundary the homotopy sphere $\Sigma_{n,p} := D^7 \cup_{\partial f^\bullet_{n,p}} D^7$

$$W_{n,p} := W_0 \cup_{f^\bullet_{n,p}} W_0.$$

By [10, Theorem p. 103], the homotopy sphere $\Sigma_{n,p}$ is determined by its Eells-Kuiper invariant which is computed by the following formula [10, (11)]:

$$\mu(\Sigma_{n,p}) := \frac{p^2_{W_{n,p}} - \sigma(W_{n,p})}{8 \cdot 28} \in \frac{1}{28}\mathbb{Z}/\mathbb{Z}$$

Here we define $p^2_{W_{n,p}} := \langle j^{-1}(p^2_{W_{n,p}}, [W_{n,p}]\rangle$ where $j \colon H^4(W_{n,p}, \Sigma_{n,p}) \cong H^4(W_{n,p})$ is the natural homomorphism. From the construction of $W_{n,p}$ we see that $H_4(W_{n,p}) \cong \mathbb{Z}(x) \oplus \mathbb{Z}(y)$ where where x is represented by the zero section of W_0 and $y = [D^4 \cup D^4]$ is represented by an embedded 4-sphere obtained by gluing two fibres of the D^4-bundle W_0 together, one from each copy of W_0. By construction, the normal bundle of the 4-sphere $D^4 \cup D^4$ has characteristic function $\gamma_{n,p}$ and hence Euler number n. It follows that the intersection form of $W_{n,p}$ with respect to the basis $\{x, y\}$ is given by the following matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$$

Moreover since x is represented by an embedded 4-sphere with normal bundle $S\alpha$ and since y is represented by an embedded 4-sphere with normal bundle $\gamma_{n,p}$ we see that for $p_W := p_{W_{n,p}}$ we have $p_W(x) = d$ and $p_W(y) = p$. We conclude that $\sigma(W_{n,p}) = 0$ and that the Poincaré dual of p_W is given by

$$PDp_{W_{n,p}} = (p - nd)x + dy.$$

It follows that $p^2_{W_{n,p}} = 2d(p - nd) + nd^2 = d(2p - nd)$ and so

$$\mu(\Sigma_{n,p}) = \frac{d(2p - nd)}{8 \cdot 28} \in \frac{1}{28}\mathbb{Z}/\mathbb{Z}. \tag{25}$$

As d is even, we see that if $8 \cdot 28$ divides $d(2p - nd)$ then $\Sigma_{n,p}$ is standard and $f_{n,p}$ extends to a diffeomorphism M_1.

In this case we shall denote any choice of extension of $f^{\bullet}_{n,p}$ by $f_{n,p}$. Since M_1 admits a unique spin structure for each orientation and since $f_{n,p}$ is orientation preserving, we see that $f_{n,p}$ is a spin diffeomorphism. Up to pseudo-isotopy, we may assume that $f_{n,p}$ is the identity on a disc and hence we may we extend $f_{n,p}$ to M by taking the connected sum with the identity on $M_2 \sharp N$. Thus we define

$$g_{n,p} := f_{n,p} \sharp \mathrm{Id}_{M_2} \sharp \mathrm{Id}_N \colon M \cong M.$$

It is clear that $g_{n,p}$ is a spin diffeomorphism and hence the mapping torus of $g_{n,p}$, $T_{g_{n,p}}$, admits a spin structure. We claim that

$$p^2_{T_{g_{n,p}}} = d(2p - nd). \tag{26}$$

This is because, as we noted above, $p^2_{T_{g_{n,p}}}$ is an invariant of the oriented bordism class of the mapping torus. It is not hard to see that there is an oriented bordism from the mapping torus $T_{g_{n,p}}$ to the disjoint union $T_{f_{n,p}} \sqcup T_{\mathrm{Id}_{M_2}} \sqcup T_{\mathrm{Id}_N}$ and the second to mapping tori make no contribution to the characteristic number. Now the mapping torus $T_{f_{n,p}}$ is oriented bordant to the twisted double

$$Y_{n,p} := W_0 \cup_{f_{n,p}} W_0$$

by the usual arguments relating mapping tori and twisted doubles. But the arguments used above to compute $p^2_{W_{n,p}}$ for $W_{n,p}$ may be repeated for Y to show that $p^2_{Y_{n,p}} = d(2p - nd)$. Hence we have

$$p^2(f_{n,p}) = p^2_{Y_{n,p}} = d(2p - nd).$$

Now recall that we may choose (n,p) freely so long as

$$(a)\ d(2p - nd) \equiv 0 \bmod 8 \cdot 28 \qquad \text{and} \qquad (b)\ n \equiv p \bmod 2. \tag{27}$$

By Lemma D_M is a homomorphism with 5.3 $D_M = -\frac{3}{28} p^2$. Hence it remains to show that we can choose (n,p) subject to the constraints above so that we have $p^2(f_{n,p}) = 8 \cdot 28 \cdot \mathrm{Num}\left(\frac{d}{112}\right)$. We therefore consider the quantity

$$\frac{p^2(f_{n,p})}{8 \cdot 28} = \frac{d(2p - nd)}{8 \cdot 28} = \frac{\mathrm{Num}\left(\frac{d}{112}\right)}{\mathrm{Denom}(\frac{d}{112})} \left(p - n\frac{d}{2}\right).$$

If $\mathrm{Denom}(\frac{d}{112})$ is even then we set $(n,p) = (0, \mathrm{Denom}(\frac{d}{112}))$. On the other hand, if $\mathrm{Denom}(\frac{d}{112})$ is odd, then 16 divides d, $\frac{d}{2}$ is even and we take $(n,p) = \left(1, \mathrm{Denom}(\frac{d}{112}) + \frac{d}{2}\right)$. Recalling that $d = d_\pi(M_0)$, this completes the proof of the lemma. □

5.5. **A conjecture on $D_M(\widetilde{\pi}_0(\mathbf{Diff}_{Spin}(M))$ for 2-connected M.** Theorem 1.13 gives a good deal of information about the difference map

$$D_M \colon \pi_0(\mathrm{Diff}_{Spin}(M)) \to \mathbb{Z}$$

and hence information about the size of $\mathcal{G}^d_2(M)$. However, determining $\mathrm{Im}(D_M)$ precisely is a subtle problem. As a first step towards solving this problem, we consider 2-connected spin 7-manifolds where by [9] there is a complete diffeomorphism classification up to connected sum with homotopy spheres. Based on this classification and other results of [9] we identify the following two cases for 2-connected 7-manifolds:

- I Manifolds M with $d_\infty(M) = d_\pi(M) = 2^k(2j+1)$ where $k \geq 3$ and the linking form of M contains an orthogonal $\mathbb{Z}_{2^k}$ summand.
- II All manifolds M not covered by Case I.

Conjecture 5.7. *For any 2-connected M with $d_\pi(M) > 0$, $\mathrm{Im}(D_M)$ is given by*

$$D_M(\widetilde{\pi}_0(\mathrm{Diff}_{Spin}(M)) = 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{224} \cdot \mathbb{Z}\right) \ \textit{in Case I,}$$

$$D_M(\widetilde{\pi}_0(\mathrm{Diff}_{Spin}(M)) = 24 \cdot \mathrm{Num}\left(\frac{d_\infty(M)}{112} \cdot \mathbb{Z}\right) \ \textit{in Case II.}$$

Remark 5.8. By Proposition 5.3, calculating $\mathrm{Im}(D_M)$ is equivalent to calculating $\mathrm{Im}(p^2)$. Now $p^2(f) = \langle p_{T_f}, [T_f]\rangle$ can also be defined for an *almost diffeomorphism* $f\colon M \cong M\sharp\Sigma$ which may be defined as a diffeomorphism from M to the connected sum of M with a homotopy sphere Σ. In this case T_f is only a piecewise linear manifold but p_{T_f} is still defined. The difference between the image of p^2 for diffeomorphisms and for almost diffeomorphisms precisely calculates the inertia group of M, $I(M)$, which is group of oriented diffeomorphism classes of homotopy spheres Σ such that there is an orientation preserving diffeomorphism $M\sharp\Sigma \cong M$. Now there are many interesting theorems about $I(M)$ for 2-connected 7-manifolds M in [25] along with interesting examples. In addition, Wilkens formulates a conjecture [25, p. 548] which computes $I(M)$ and Conjecture 5.7 is closely related to Wilkens' conjecture.

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